

Chapter 1

When scientific, engineering, business problem statistics, economic arrive of the mathematical formulas of the problem sometimes these mathematical problems could not be solved by any analytical methods at that time, we need numerical method.

Analytical method may not be exist to solve analytical methods but are quite time consuming due to huge data functions involved.

Mathematical models are a ^{central} piece of science and engineering. Some models have closed form solution, i.e. they can be solved analytically. Many model cannot be solved analytically or the analytic solution is too costly to be practical. All models can be solved computationally and the result may not be the exact answer but it can be useful. Since, errors are inevitable a ~~form~~ more practical question we should to ask is how close the answer needs to be for it to be useful. In lots of engineering situation, the only way to get the right answer is to be use the following formula.

Right Answer = Wrong answer + corrections.

Error :

Error is the measure of the estimated difference between the observed or calculated value of a quantity and its true value.

Error in numerical computations :

In any numerical ~~qua~~ analysis, error will arise during calculations. To be able to deal with the user of error, we need to identify where the error is coming from followed by quantifying the error, and lastly minimize the error as per our needs.

Types of error :

There are four types of errors :

- ① True errors
- ② Relative errors
- ③ Approximate errors
- ④ Relative Approximate errors

① True Error :

It is denoted by E_t . It is the difference between the true value (exact value) and approximate value.

$$\text{True } \overset{\text{error}}{\cancel{\text{error}}} = \text{True Value} - \text{Approximate Value}.$$

The derivative of function $f(x)$ at a particular value can be approximately calculated by,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Q For $f(x) = 7e^{0.5x}$ and $h=0.3$, find the approximate value of $f'(2)$, the value of $f'(2)$, and the true error.

Solⁿ:

Approximate value of derivation is calculated as,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

now,

$$\text{for } x=2, h=0.3$$

$$f'(2) = \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5 \times 2.3} - 7e^{0.5 \times 2}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3}$$

$$= 10.265$$

The exact value of $f'(2)$ can be calculated as using differential calculus,

$$f(x) = 7e^{0.5x}$$

$$f'(x) = 7 \times 0.5 \times e^{0.5x}$$

So, true value of $f'(2)$ is

$$f'(2) = 7 \times 0.5 \times e^{0.5 \times 2}$$

$$= 9.5140$$

now,

True error is calculated as,

$$\text{True Error, } (E_t) = \text{True Value} - \text{Approximate Value}$$

$$= 9.5140 - 10.265$$

$$= -0.75061$$

② Relative Errors:

Relative true error is denoted by E_r and is defined as the ratio between the true error and the true value.

$$\text{Relative True error, } E_r = \frac{\text{True Error}}{\text{True Value}}$$

Q

For $f(x) = 7e^{0.5x}$ and $h = 0.3$, find the relative true error at $x = 2$ and 3 .

Approximate value of derivation is calculated as

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Now,

for $x=2$ and $h=0.3$

$$f'(2) = \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7 \times e^{0.5 \times 2.3} - 7 \times e^{0.5 \times 2}}{0.3}$$

$$= 10.265$$

The true value of $f'(2)$ can be calculated as using differential calculus,

$$f(x) = 7e^{0.5x}$$

$$f'(x) = 7 \times 0.5 e^{0.5x}$$

So, true value of $f'(2)$ is

$$\begin{aligned} f'(2) &= 7 \times 0.5 e^{0.5 \times 2} \\ &= 9.5140 \end{aligned}$$

Now,

True error is calculated as,

$$\begin{aligned}\text{True error} &= \text{True Value} - \text{Approximate value} \\ &= 9.5140 - 10.265 \\ &= -0.75061\end{aligned}$$

Then,

$$\text{Relative error} = \frac{\text{True error}}{\text{True value}}$$

$$= \frac{-0.75061}{9.5140}$$

$$= -0.078895$$

$$= -0.078895 \times 100\%$$

$$= -7.8895\%$$

~~1.8895~~

$$|Er| = 7.8895$$

Then for, $n = 3$, $h = 0.3$

$$f'(3) = \frac{f(3+0.3) - f(3)}{0.3}$$

$$= \frac{f(3.3) - f(3)}{0.3}$$

$$= \frac{7 * e^{0.5(3.3)} - 7 * e^{0.5(3)}}{0.3}$$

=

The true value of $f'(3)$ can be calculated as ~~diff~~ using different calculus.

True value of $f'(3)$ is,

$$f'(3) = 7 * 0.5 * e^{0.5 * 3}$$

=

Now,

True error = True Value - Approximate Value

=

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Then,

Relative Error, $E_r = \frac{\text{True Error}}{\text{True value}}$

③ Approximate Errors :-

The approximate error is denoted by E_a and is defined as the difference between the present approximation and the previous approximation.

Approximate Error, $E_a = \text{present approximation} - \text{previous approximation}$.

Q

$f(x) = 7e^{0.5x}$ and $x=2$, find the following

a) $f'(2)$ using $h=0.3$

b) $f'(2)$ using $h=0.15$

c) Approximate error for the value of $f'(2)$.

Solⁿ.

The approximate expression for the derivative of a function is,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

for $x=2$, $h=0.3$

$$f'(2) = \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7 * e^{0.5(2.3)} - 7 * e^{0.5(2)}}{0.3}$$

$$= 10.265$$

Again, at $x=2$, $h=0.15$

$$f'(2) = \frac{f(2+0.15) - f(2)}{0.15}$$

$$= \frac{f(2.15) - f(2)}{0.15}$$

$$= \frac{7 * e^{0.5(2.15)} - 7 * e^{0.5(2)}}{0.15}$$

$$= 9.8799$$

$$\text{Approximate error, } E_a = 9.8799 - 10.265$$

$$= -0.38474$$

4) Relative Approximate error:

It is denoted by E_{ra} and is defined as the ratio between approximate error and the present approximation.

$$E_{ra} = \frac{\text{Approximate error}}{\text{present approximation}}$$

Q. $f(x) = 7e^{0.5}$, find the Relative approximate error in calculation $f'(2)$ using values from $h=0.3$ & 0.15 .

Sol.

$$E_{ra} = \frac{\text{Approximate error}}{\text{present approximation}}$$

$$= \frac{-0.3847}{9.879}$$

$$= -0.0389$$

$$|E_{ra}| = 0.0389$$

E_{ra} also be presented as percentages.

$$E_{ra} = 0.0389 \times 100\%$$

$$= 3.89\%$$

Source of errors:

- Mistakes in program.
- Mistakes in the measurement of physical quantities.

In Numerical method two errors we need to focus:-

1) Round off error:-

The computer can only represent a number approximately.

eg: A number $\frac{1}{3}$ may be represented as 0.3333 on the PC then the round off error in this case is $\frac{1}{3} - 0.3333 = 0.000033$

2) Truncation error:-

Truncation error is defined as the error caused by truncating a mathematical procedure. Maclaurin series for e^x is given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

When we need to calculate e^x , only a finite number of terms ^{can be} used. If

one uses n term to calculate e^x then

$$e^x = 1 + x + \frac{x^2}{2!}$$

Truncation error:

$$= e^x - (1 + x + \frac{x^2}{2!})$$

$$= \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Taylor's Theorem:

It is given by,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots$$

If all the derivative of $f(x)$ exist and are continuous between x and $x+h$. Equation (i) can be written as,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^{(n)}(x)}{n!}h^n + \dots$$

If we put $h=0$ in equation (ii), we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

eg $f(x) = e^x$ is called Maclaurian's Series.

Q. Find the value of $e^{0.25}$ using the first five terms of the Maclurin series.

Solⁿ:

The five terms of the Maclurin series for e^x can be calculated as,

$$\begin{aligned}f(x) &= e^x &\Rightarrow f(0) &= 1 \\f'(x) &= e^x &\Rightarrow f'(0) &= 1 \\f''(x) &= e^x &\Rightarrow f''(0) &= 1 \\f'''(x) &= e^x &\Rightarrow f'''(0) &= 1 \\f^{(4)}(x) &= e^x &\Rightarrow f^{(4)}(0) &= 1\end{aligned}$$

Thus, from Maclurin series,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \dots$$

$$= 1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^4}{4!}$$

$$= 1 + 0.25 + \frac{0.25^2}{2!} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$= 1.2840$$

The exact value of $e^{0.25}$ upto 5 significant digits is also 1.2840.

Solving non-linear equation:

Equation whose graph does not form a straight line (linear) is called non-linear equation. In a non-linear equation, the variables are either of degree greater than 1 or less than 1 but never 1.

general form,

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0$$

where, a

$a_n, a_{n-1}, \dots, a_0$ are known as numbers

and we are trying to find the value of x .

The solution process often involves finding the value of x that would satisfy the equation. These values are called roots or zeros of the function $f(x)$. $f(x)$ is said to be non-linear, if the response of the dependent variable y is not in direct or exact proportion to the changes in the independent variable x .

eg:

$$y = x^2 + 1$$

Trail and error method :

In this method we can make a series of guess for x and then evaluate $f(x)$ at that x . If it is closed to 0, it is one of the approximate error (Root) of the given non-linear equation. Otherwise, make another guesses for x repeat the same process again until the x for which $f(x)=0$ is found.

Q

Solve the equation $f(x) = 2x^2 + x - 6 = 0$

Solⁿ:

Given

$$x=0, x=1, x=2, x=1.5, x=1, x=-2$$

put,

$$x=0$$

$$f(x) = -6$$

put,

$$x=-1$$

$$f(x) = -5$$

put,

$$x=1$$

$$f(x) = -3$$

put,

$$x=-2$$

$$f(x) = 0$$

put,

$$x=2$$

$$f(x) = 4$$

put,

$$x=1.5$$

$$f(x) = 0$$

Iterative method:

An iterative method starts with an initial approximate value of the root, known as initial ~~guess~~ guess and then find ^{more and more} correct approximate solutions in successive iteration.

- i) Bisection method
- ii) Newton Raphson's method
- iii) Secant method
- iv) Fixed point method.

i) Bisection method (Half-interval method)

In equation $f(x) = 0$ where, $f(x)$ is real and continuous function has at least one root between A and B, if $f(a) \cdot f(b) < 0$ i.e. $f(a) \cdot f(b)$ are of opposite sign.

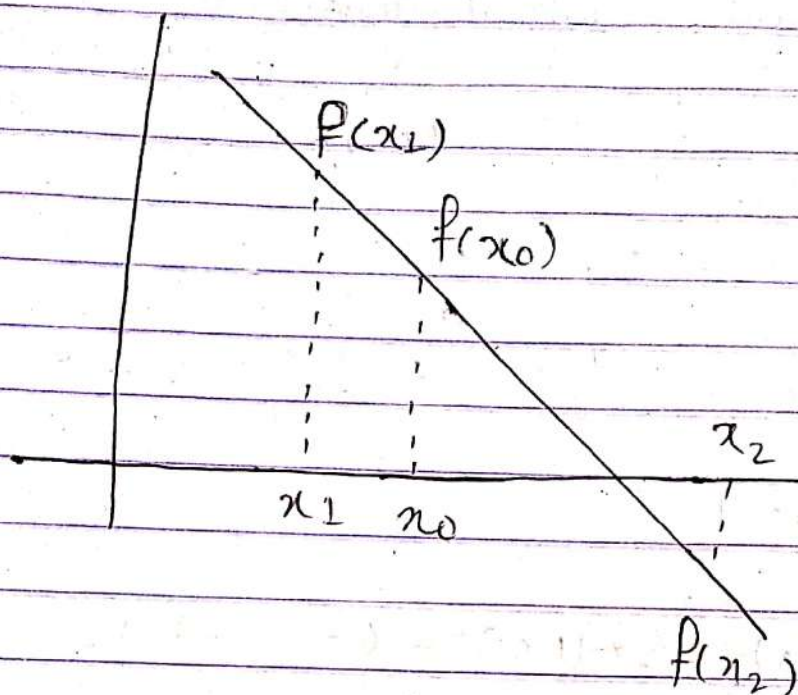
Let $x_1 = a$ and $x_2 = b$

Let $x_1 = a$, $x_2 = b$. Let us also define another point x_0 to be the mid-point between A and B i.e. $x_0 = \frac{x_1 + x_2}{2}$.

Now, there exist following three conditions:

- a) If $f(x_0) = 0$, we have root at x_0 .
- b) If $f(x_0) \cdot f(x_1) < 0$, there is a root between x_0 and x_1 .

(i) If $f(x_1) \cdot f(x_2) < 0$, there is a root between x_1 and x_2 .



Algorithm :

1. Decide initial values for x_1 and x_2 and stopping criterion ϵ .
2. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$.
3. If f_1 and f_2 ($f_1 \cdot f_2$) > 0 , x_1 and x_2 do not bracket any root and go to step 7.
4. Compute $x_0 = (x_1 + x_2) / 2$ and compute $f_0 = f(x_0)$.
5. If $f_1 \cdot f_2 < 0$ then set $x_2 = x_0$ else set $x_1 = x_0$.
6. If absolute value of $(x_2 - x_1) / x_2 \leq \epsilon$ then root = $(x_1 + x_2) / 2$ write the value of root and go to step 7. Else go to step 4.
7. Stop.

$$Q \quad f(x) \Rightarrow 3x^2 - 6x + 2 = 0$$

Solⁿ :

Assume initial guess are $x_1 = 1$ and $x_2 = 2$

Iteration - 1 :

$$f(x_1) = -1$$

$$f(x_2) = 2$$

$$x_0 = \frac{x_1 + x_2}{2} = \frac{1 + 2}{2} = 1.5$$

$$x_0 = 1.5$$

$$\begin{aligned} f(x_0) &= 3 \times (1.5)^2 - 6 \times (1.5) + 2 \\ &= -0.25 \end{aligned}$$

$$\begin{aligned} \text{Error, } E &= \frac{2 - 1.5}{2} \\ &= 0.25 \end{aligned}$$

Iteration - 2 :

$$x_1 = 1.5$$

$$, x_2 = 2$$

$$f(x_1) = -0.25$$

$$, f(x_2) = 2$$

$$x_0 = \frac{x_1 + x_2}{2} = \frac{1.5 + 2}{2} = 1.75$$

$$f(x_0) = 0.6875$$

$$\text{Error, } E = \frac{1.75 - 1.5}{1.75} = 0.2428$$

Iteration - 3 :

$$x_1 = 1.5, \quad x_2 = 1.75$$

$$f(x_1) = -0.25$$

$$f(x_2) = 0.6875$$

$$x_0 = \frac{x_1 + x_2}{2} = \frac{-0.25 + 0.6875}{2} = 0.4375$$

$$\text{Error, } E = \frac{0.6875 - 0.4375}{0.6875} = \frac{x_2 - x_0}{x_2}$$

$$= 0.2428 \quad 0.3636363636$$

$$f(x_0) = -0.05078125$$

Iteration - 4 :

$$x_1 = 0.4375, \quad x_2 = 1.75$$

$$f(x_1) = -0.05078125$$

$$f(x_2) = 0.6875$$

$$x_0 = \frac{-0.05078125 + 0.6875}{2} = 0.328359375$$

$$f(x_0) = 0.393901825$$

$$\text{Error, } E = \frac{x_2 - x_0}{x_2} = \frac{0.6875 - 0.318359375}{0.6875} \\ = 0.5369318182$$

Iteration, -5 :

$$x_1 = 0.4375, \quad x_2 = \cancel{1.4} 0.318359375$$

$$f(x_1) = -0.05078125$$

$$f(x_2) = 0.393901825$$

$$x_0 = \frac{0.4375 + 0.318359375}{2}$$

$$= 0.3779296875$$

$$f(x_0) = 0.1609144211$$

$$\text{Error, } E = \frac{x_2 - x_0}{x_2}$$

$$= \frac{0.393901825 - 0.3779296875}{0.393901825}$$

$$= 0.04054852362$$

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Convergence of bisection method:

In bisection method, the interval containing the root is reduced by a factor of two. The same process is repeated for new interval. If the procedure is repeated for n times, then the interval containing the root is reduced to the size, $\frac{x_2 - x_1}{2^n} = \frac{\Delta x}{2^n}$

Now, we say that the maximum error after n^{th} iteration,

$$E_n = \frac{x_2 - x_1}{2^n}$$

Similarly, after $(n+1)^{\text{th}}$ iteration maximum error is given by,

$$E_{n+1} = \frac{x_2 - x_1}{2^{n+1}}$$

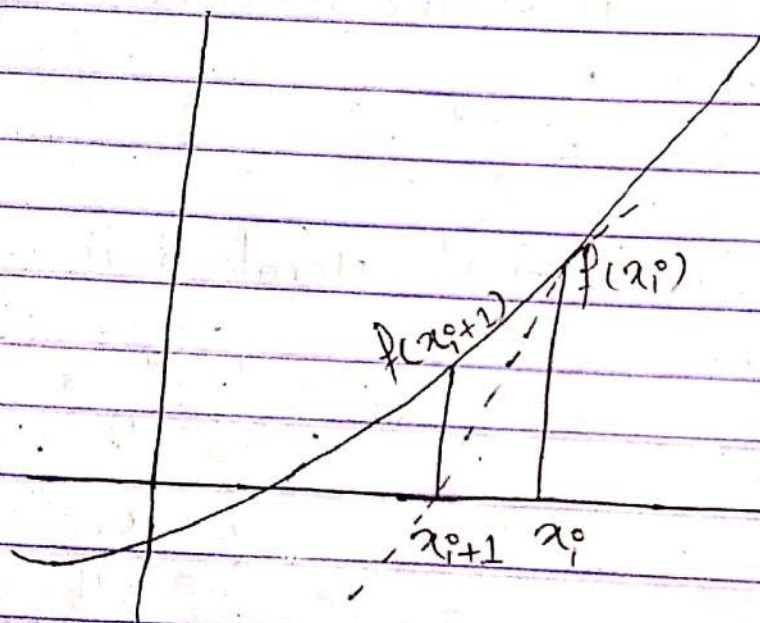
$$= \frac{x_2 - x_1}{2^n \cdot 2}$$

$$E_{n+1} = \frac{E_n}{2}$$

The ^{equation} shows that the error is halved after each iteration of bisection method. Thus bisection method converges.

ii) Newton Raphson Method :

Newton Raphson (NR) method is based on the principle that if the initial guess of the root of $f(x) = 0$ at x_i , and one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is the improved estimate of the root.



Using the definition of the slope of a function at $x = x_i$

$$f'(x) = \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}}$$

If $f(x)$ have the root at x_{i+1} , then $f(x_{i+1}) = 0$.
Putting this in above eqⁿ, we get

$$f'(x) = \frac{f(x_0) - f(x_{p+1})}{x_0 - x_{p+1}}$$

This gives,

$$x_{i+1} = x_i - \frac{f(x)}{f'(x)}$$

This eqn is called Newton Raphson method.

Algorithms :

- ① start
- ② Input initial guess x_0 at precision E .
- ③ Evaluate $f'(x_0)$ symbolically.
- ④ Use x_0 to estimate the new value of the root x_1 as,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- ⑤ Find the absolute relative approximate error,

$$|E_a| = \left| \frac{x_1 - x_0}{x_1} \right|$$

- ⑥ Compare the absolute relative error with pre-specified relative error tolerance

$$\text{If } |E_a| > E,$$

$$x_0 = x_2$$

else goto step 3.

goto step 7.

iii) stop.

Q Solve the eqⁿ $x^2 + 4x - 9 = 0$ by using Newton Raphson. Assume error precision is 0.1.

Solⁿ:

$$f(x) = x^2 + 4x - 9$$

$$f'(x) = 2x + 4$$

Let us assume that initial guess value is 1.

1st iteration,

$$x_0 = 1$$

$$f(x_0) = -4$$

$$f'(x_0) = 6$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\begin{aligned}
 &= 1 - \frac{-4}{6} \\
 &= \frac{6+4}{6} \\
 &= \frac{10}{6} \\
 &= 1.667
 \end{aligned}$$

$$\begin{aligned}
 E_a &= \frac{x_1 - x_0}{x_1} \\
 &= \frac{1.667 - 1}{1.667} \\
 &= 0.40019378
 \end{aligned}$$

2nd Iteration,

$$\begin{aligned}
 x_0 &= 1.667 \\
 f(x_0) &= 0.447 \\
 f'(x_0) &= 7.334
 \end{aligned}$$

$$\begin{aligned}
 x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
 &= 1.667 - \frac{0.447}{7.334} \\
 &= 1.606
 \end{aligned}$$

$$\begin{aligned}
 E_a &= \frac{x_1 - x_0}{x_1} \\
 &= \frac{1.606 - 1.667}{1.606} \\
 &= 0.04122423481
 \end{aligned}$$

3rd Iteration,

$$\begin{aligned}
 x_0 &= 1.606 \\
 f(x_0) &= 0.003 \\
 f'(x_0) &= 7.222
 \end{aligned}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.606 - \frac{0.003}{7.212}$$

$$= 1.6055$$

$$E_a = \left| \frac{1.6055 - 1.606}{1.6055} \right|$$

$$= 0.00031142$$

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Convergence of NRM :

Suppose, $f(x)$ is a function having exact root ' α '.

$$f(\alpha) = 0$$

Suppose x_n is an approximate root of function $f(x)$.

$$x_n = \alpha + e_n$$

where, $e_n = \text{error}$.

$$x_{n+1} = \alpha + e_{n+1}$$

We have NR formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n+1})}$

$$\alpha + e_{n+1} = \alpha + e_n - \frac{f(x_n)}{f'(x_{n+1})}$$

$$e_{n+1} = e_n - \frac{f(x_n + \alpha)}{f'(x_n + \alpha)}$$

Applying Taylor Theorem:

$$e_{n+1} = e_n - \frac{f(\alpha) + f'(\alpha)e_n + \frac{f''(\alpha)}{2!}e_n^2 + \dots}{f'(\alpha) + f''(\alpha)e_n + \frac{f'''(\alpha)}{2!}e_n^2 + \dots}$$

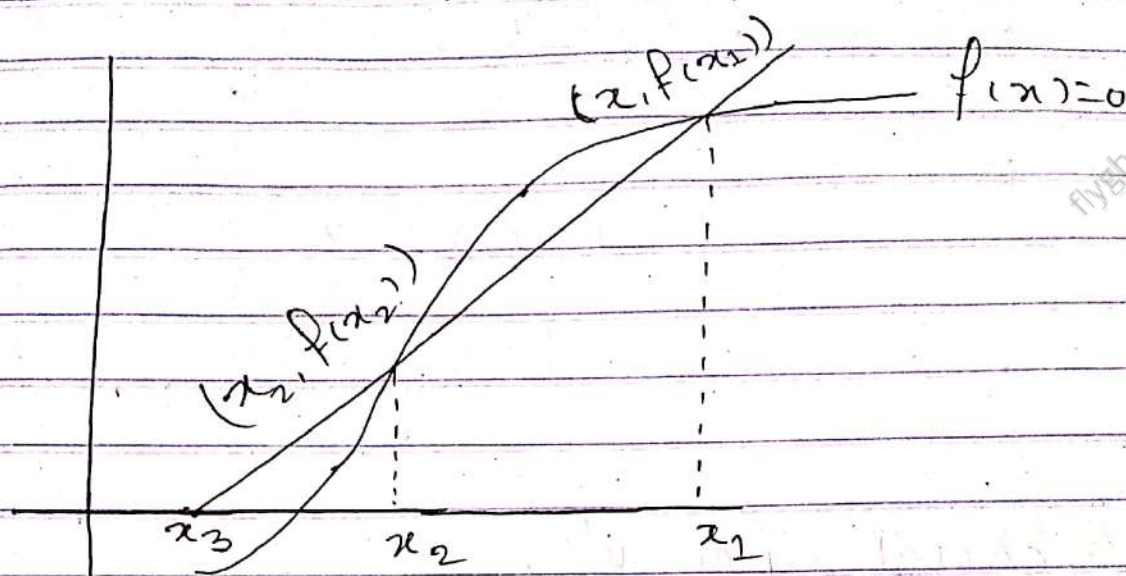
$$= \frac{e_n f'(\alpha) + e_n^2 f''(\alpha) + \frac{e_n^3 f'''(\alpha)}{2!} + \dots}{f'(\alpha) + e_n f''(\alpha) + \dots} - \frac{f(\alpha) + f'(\alpha)e_n + \frac{f''(\alpha)e_n^2}{2!} + \dots}{f'(\alpha) + f''(\alpha)e_n + \frac{f'''(\alpha)e_n^2}{2!} + \dots}$$

$$= \frac{1/2 e_n^2 f''(\alpha)}{f'(\alpha)}$$

$$e_{n+1} = \frac{f''(\alpha) e_n^2}{2f'(\alpha)}$$

$$e_{n+1} \propto e_n^2$$

(iii) Secant Method :



Secant method is the line joining two points of a curve.

Consider, a curve $f(x)=0$ as shown in figure. Let x_1, x_2 be initial points slope of the secant line is given by,

$$\text{or, } \frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

$$\text{or, } f(x_1) \cdot (x_2 - x_3) = f(x_2) \cdot (x_1 - x_3)$$

$$\text{or, } f(x_1) \cdot x_2 - x_3 \cdot f(x_1) = f(x_2) \cdot x_1 - x_3 \cdot f(x_2)$$

$$\text{or, } x_3 \{ f(x_2) - f(x_1) \} = f(x_2) \cdot \{ x_1 - f(x_1) \cdot x_2 \}$$

$$\text{or, } x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

$$\text{or, } x_3 = x_2 - \frac{f(x_2) * (x_2 - x_1)}{f(x_2) - f(x_1)}$$

General Formula,

$$x_{n+1} = x_n - \frac{f(x_n) * (x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Algorithm

i) Start

ii) Input two initial guesses (x_0 & x_1) and precision ϵ .

iii) Evaluate $f(x_0)$ & $f(x_1)$

iv) Estimate new value of the root x_2 as,

$$x_2 = x_1 - \frac{f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)}$$

v) find the absolute relative approximate error $|E_a|$ as,

$$|E_a| = \frac{x_2 - x_1}{x_2}$$

vi) Compare the $|E_a|$ with the pre-specified relative error E .

if

$$|E_a| < E$$

then

Stop.

else

$$|E_a| > E$$

$$x_0 = x_1$$

$$x_1 = x_2$$

$$f(x_0) = f(x_1)$$

goto step 3.

else if

goto step 7

vii) Stop.

Q Solve the eqⁿ $2x^2 + 4x - 10 = 0$ using secant method with error precision 0.05.

Solⁿ:

Roots are,

$$x_0 = 1.4$$

$$x_1 = 1.5$$

1st Iteration,

$$f(x_0) = -0.48$$

$$f(x_1) = 0.5$$

$$x_2 = x_1 - \frac{f(x_1) \times (x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1.5 - \frac{0.5 \times (1.5 - 1.4)}{0.5 - (-0.48)}$$

$$= 1.4489$$

$$\text{Error, } E = \frac{x_2 - x_1}{x_2} \Rightarrow \left| \frac{1.4489 - 1.5}{1.4489} \right|$$
$$\geq 0.03521$$

2nd Iteration,

$$x_0 = 1.5, x_1 = 1.4489$$

$$f(x_0) = 0.5, f(x_1) = -0.0049979$$

$$x_2 = x_1 - \frac{f(x_1) \times (x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1.4489 - \frac{0.0049979 \times (1.4489 - 1.5)}{0.0049979 - (0.5)}$$

$$= 1.449484$$

$$\text{Error, } E = \left| \frac{1.449484 - 1.4489}{1.449484} \right| = 0.0039836 //$$

(Pv) Fixed point Iteration Method :-

Here we rearrange the function $f(x)=0$ such that x is on the left-hand side of the equation.
i.e.

$$f(x)=0 \text{ --- (i)}$$

can be written as $x=g(x)$ --- (ii)

eqⁿ (i) can be written in the form of eqⁿ (ii) either by algebraic manipulation or by adding x to both sides of eqⁿ.
for example :

$x^2 - x - 2 = 0$ can be written as $x = x^2 - 2$,
and eqⁿ $\tan x = 0$ can be written as $x = x + \tan x$.

Eqⁿ (i) and eqⁿ (ii) are equivalent, so a root of equation (ii) is also a root of equation (i). The root of the equation (ii) is the point of intersection of a straight line $y=x$ and the curve $x=g(x)$. This intersection point is called fixed point of $g(x)$. The equation (ii) is called fixed point equation. It provides a convenient form of predicting the value of x as a function of x .

If x_0 is the initial guess to a root then the next approximation is given by,

$$x_1 = g(x_0)$$

Similarly, next approximation is,

$$x_2 = g(x_1)$$

In general,

$$x_{n+1} = g(x_n)$$

This is called fixed point iteration method.

Algorithm:

(i) Start

(ii) Input initial guess (say x_0) and precision (say E)

(iii) Convert $f(x) = 0$ to the form $x = g(x)$.

(iv) Estimate new value of the root x_1 as,
$$x_1 = g(x_0)$$

(v) Find the absolute relative approximate error,

$$|E_a| = \left| \frac{x_1 - x_0}{x_1} \right|$$

(vi) Compare the error $|E_a|$ with the pre-specified relative error tolerance E .

If

$$|E_a| > E$$

else

$$x_0 = x_1$$

goto step 7

goto step 4

(vii) stop.

Q solve the equation $x^2 = 6x + 820$ by assuming error precision 0.025

Solⁿ:

given function,

$$f(x) = x^2 - 6x + 8$$

$$f(x) = 0$$

$$x^2 - 6x + 8 = 0$$

$$x = \frac{x^2 + 8}{6}$$

$$g(x) = \frac{x^2 + 8}{6}$$

1st iteration,

initial root,

$$x_0 = 1$$

$$x_1 = g(x_0)$$

$$= \frac{1^2 + 8}{6} = 1.5$$

$$|E_a| = \left| \frac{x_1 - x_0}{x_1} \right|$$

$$= \left| \frac{1.5 - 1}{1.5} \right| = 0.33333$$

2nd iteration,

$$x_0 = 1.5$$

$$x_1 = \frac{(1.5)^2 + 8}{6} = 1.70833$$

$$|E_n| = \left| \frac{1.70833 - 1.5}{1.70833} \right|$$

$$= 0.121949$$

3rd iteration,

$$x_0 = 1.70833$$

$$x_1 = \frac{(1.70833)^2 + 8}{6} = 1.819732$$

$$|E_n| = \left| \frac{1.819732 - 1.70833}{1.819732} \right| = 0.0612183$$

4th iteration,

$$x_0 = 1.819732$$

$$x_1 = \frac{(1.819732)^2 + 8}{6} = 1.885236819$$

$$|E_n| = \frac{1.885236819 - 1.819732}{1.88523682}$$

$$= 0.0347467$$

5th iteration,

$$x_0 = 1.885236819$$

$$x_1 = \frac{(1.885236819)^2 + 8}{6} = 1.925686322$$

$$|E_n| = \frac{1.925686322 - 1.885236819}{1.925686322}$$

$$= 0.0210052$$

Unit 2

Interpolation :

Interpolation :

Normally a function $y = f(x)$ is given only at discrete point such as $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$.

And we need to find the value of y at any other value of x . The process of computing the values of y for given x inside the given range is called interpolation.

And the process of computing the values of y for given x outside the range is called extrapolation.

$x < x_0$ or $x > x_n$ then it is called extrapolation.

Lagrange's Interpolation :

Let us consider a second order polynomial of the form: $P_2(x) = b_1(x - x_0) + b_2(x - x_1) + b_3(x - x_2)$

$$P_2(x) = b_1(x - x_0) + b_2(x - x_1) + b_3(x - x_2) \quad \text{--- (i)}$$

Let (x_0, b_0) , (x_1, b_1) and (x_2, b_2) are three interpolating points, then we have,

$$P_2(x_0) = b_2(x_0 - x_1)(x_0 - x_2) = f_0$$

$$\Rightarrow b_2 = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)}$$

$$(x_0 - x_1)(x_0 - x_2)$$

$$P_2(x_1) = b_3(x_1 - x_2)(x_1 - x_0) = f_1$$

$$\Rightarrow b_3 = \frac{f_1}{(x_1 - x_2)(x_1 - x_0)}$$

$$P_2(x_2) = b_2(x_2 - x_0)(x_2 - x_1) = f_2$$

$$\Rightarrow b_2 = \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$$

Substituting those values of b_1, b_2, b_3 in eq (i), we get

$$P_2(x) = \frac{f_2(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f_0(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1(x - x_2)(x - x_0)}{(x_1 - x_2)(x_1 - x_0)}$$

$$= \frac{f_0(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1(x - x_2)(x - x_0)}{(x_1 - x_2)(x_1 - x_0)} +$$

$$\frac{f_2(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Above equation can be written as,

$$P_2(x) = f_0l_0 + f_1l_1 + f_2l_2 = \sum_{i=0}^2 f_i l_i(x) \quad (2)$$

Generalizing eqn (2) for $n+1$ points, we get the relation,

$$P_n(x) = \sum_{i=0}^n f_i l_i(x) \quad (3)$$

here,

$$l_i(x) = \prod_{j=0, j \neq i}^n \frac{(x - x_j)}{(x_i - x_j)}$$

eqn (3) is called

Lagrange interpolation.

Q The table below gives square roots of integer

x	2	2.5	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

question incomplete but $x = 2.5$.

$$x_0 = 2, \quad x_1 = 3, \quad x_2 = 4$$

$$x = 2.5$$

$$f_0 = 1.4142, \quad f_1 = 1.7321, \quad f_2 = 2$$

$$P_2(x) = \frac{f_0(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{f_1(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + \frac{f_2(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{1.4142(2.5-3)(2.5-4)}{(2-3)(2-4)} + \frac{1.7321(2.5-2)(2.5-4)}{(3-2)(3-4)} + \frac{2(2.5-2)(2.5-3)}{(4-2)(4-3)}$$

$$= 0.530325 + 1.299075 + 0.25$$

$$= 1.5799$$

//

time t	0	10	15	20	22.5	30
velocity V	0	227.04	362.78	517.35	602.97	701.67

Determine the value of following velocity at $t=16$ seconds using a first order Lagrange interpolation.

sol

$$x_0 = 15, \quad x_1 = 20, \quad x = 16$$

$$f_0 = 362.78, \quad f_1 = 517.35$$

$$P_1(x) = \frac{f_0(x-x_1)}{(x_0-x_1)} + \frac{f_1(x-x_0)}{(x_1-x_0)}$$

$$= \frac{362.78(16-20)}{(15-20)} + \frac{517.35(16-15)}{(20-15)}$$

$$= 290.224 + 203.47$$

$$= 393.694$$

//

9. The value of e^x is given below:

x	0	1	2	20
e^x	1	2.7183	7.3891	20.0855

Determine the value of $e^{1.2}$ by using second order polynomial interpolation using Lagrange polynomial.

sol.

$$x = 1.2, \quad x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$$

$$f_0 = 1, \quad f_1 = 2.7183, \quad f_2 = 7.3891$$

Lagrange polynomial second order polynomial interpolation is,

$$P_2(x) = \frac{f_0 (x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{f_1 (x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} +$$

$$\frac{f_2 (x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{1 * (1.2-1)(1.2-2)}{(0-1)(0-2)} + \frac{2.7183(1.2-0)(1.2-2)}{(1-0)(1-2)} +$$

$$\frac{7.3891(1.2-0)(1.2-1)}{(2-0)(2-1)}$$

$$= -0.08 + 2.609568 + 0.886692$$

$$= 3.41626 //$$

Q	time t	0	10	15	20	22.5	30
	velocity V	0	227.04	362.78	517.35	602.97	901.67

Determine the value of velocity at $t = 16$ using a fourth order Lagrange interpolation.

Solⁿ:

$$x = 16, x_0 = 0, x_1 = 10, x_2 = 15, x_3 = 20, x_4 = 22.5$$

$$f_0 = 0, f_1 = 227.04, f_2 = 362.78, f_3 = 517.35, f_4 = 602.97$$

$$x_0 = 10, x_1 = 15, x_2 = 20, x_3 = 22.5, x_4 = 30$$

$$x = 16$$

$$f_0 = 227.04, f_1 = 362.78, f_2 = 517.35, f_3 = 602.97, f_4 = 901.67$$

The fourth order Lagrange interpolation is,

$$P_4(x) = \frac{f_0(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} +$$

$$\frac{f_1(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} +$$

$$\frac{f_2(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} +$$

$$\frac{f_3(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} +$$

$$\frac{f_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$= \frac{227.04(16-15)(16-20)(16-22.5)(16-30)}{(10-15)(10-20)(10-22.5)(10-30)} +$$

$$\frac{362.78(16-10)(16-20)(16-22.5)(16-30)}{(15-10)(15-20)(15-22.5)(15-30)} +$$

$$\frac{517.35(16-10)(16-15)(16-22.5)(16-30)}{(20-10)(20-15)(20-22.5)(20-30)} +$$

$$\frac{602.97(16-10)(16-15)(16-20)(16-30)}{(22.5-10)(22.5-15)(22.5-20)(22.5-30)} +$$

$$\frac{901.67 (16-10) (16-15) (16-20) (16-22.5)}{(30-10) (30-15) (30-20) (30-22.5)}$$

$$= -6.6114048 + 281.7407627 + 225.97848 - 115.2557051 + 6.25157867$$

$$= 622.5851221$$

//

Newton's divide difference interpolation!

Let us consider a polynomial of degree n of the form given below:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})$$

To construct the polynomial we need to find coefficients a_0, a_1, a_2 and a_n . Let us suppose $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$ are given interpolating points.

Now, at $x=x_0$ equation (1) becomes.

$$p(x_0) = f(x_0) = a_0$$

$$\Rightarrow a_0 = f(x_0)$$

at $x=x_1$ equation (1) becomes,

$$p(x_1) = f(x_1) = a_0 + a_1(x_1 - x_0)$$

$$\Rightarrow a_1 = \frac{f(x_1) - a_0}{(x_1 - x_0)} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

at $x=x_2$ equation (1) becomes,

$$p(x_2) = f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow a_2 = \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$\Rightarrow a_2 = \frac{f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)(x_2 - x_0)}{(x_1 - x_0)}}{(x_2 - x_0)(x_2 - x_1)}$$

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that,

a_0 , a_1 and a_2 are finite divided differences. a_0 , a_1 and a_2 are the first, second and third finite divided differences respectively. We denote the first divided differences by,

$$f[x_0] = f(x_0)$$

The second divided difference by,

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The third divided difference by,

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

This leads us to writing the general form of the newton's divide difference polynomial for $n+1$ data points, (x_0, y_0) , (x_1, y_1) , ..., (x_{n-1}, y_{n-1}) , (x_n, y_n) as

$$p_n(x) = a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

where,

$$a_0 = f(x_0)$$

$$a_1 = f[x_1, x_0]$$

$$a_2 = f[x_2, x_1, x_0]$$

...

$$a_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_1, x_0]$$

$$a_n = f[x_n, x_{n-1}, \dots, x_1, x_0]$$

Q

Time (t)	0	10	15	20	22.5
Velocity (V)	0	227.04	362.78	517.35	602.97
				30	
				901.67	

Determine the value of the velocity at $t=16$ seconds with third order polynomial using Newton's divide difference polynomial method.

Solⁿ: for a third order polynomial, the velocity is given by,

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$x=16, \quad x_0=10, \quad x_1=15, \quad x_2=20, \quad x_3=22.5$$

$$f_0=227.04, \quad f_1=362.78, \quad f_2=517.35, \quad f_3=602.97$$

$$a_0 = f(x_0) = 227.04$$

$$a_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$a_2 = f[x_2, x_1, x_0]$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{\frac{517.35 - 361.78}{20 - 15} - 27.148}{20 - 10}$$

$$= 0.3766$$

$$a_3 = f[x_3, x_2, x_1, x_0]$$

$$= \frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{\frac{602.97 - 517.35}{22.5 - 20} - \frac{517.35 - 361.78}{20 - 15} - 27.148}{22.5 - 10}$$

$$= 0.00543$$

Now,

The third order polynomial can be written as,

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= 227.04 + 27.148(16-10) + 0.3766(16-10)(16-15) + 0.00543(16-10)(16-15)(16-20)$$

$$= 392.05728$$

Q. Given the following data points, obtain the table of divided differences. Use the table to find third order Lagrange polynomial and use the polynomial to estimate the value of $f(8)$.

x	3	7	9	10
$f(x)$	168	120	72	63

for a third order Lagrange polynomial,

$x(28)$

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$x_0 = 3, x_1 = 7, x_2 = 9, x_3 = 10$$

$$f_0 = 168, f_1 = 120, f_2 = 72, f_3 = 63$$

$$a_0 = f(x_0) = 168$$

$$a_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{120 - 168}{7 - 3}$$

$$= -12$$

$$a_2 = f[x_2, x_1, x_0]$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{72 - 120}{9 - 7} - (-12)}{9 - 3}$$

$$= -2$$

$$a_3 = f[x_3, x_2, x_1, x_0]$$

$$= \frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - f[x_2, x_1, x_0]}{x_3 - x_0}$$

$$= \frac{\frac{63 - 72}{10 - 9} - (-2)}{10 - 3}$$

$$= -1$$

Now,

Third polynomial of order polynomial can be written as,

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= 168 + (-12)(x-3) + (-2)(x-3)(x-7) + (-1)(x-3)(x-7)(x-9)$$

$$= 168 - 12x + 36 - 2x^2 + 20x - 42 + x^3 - 19x^2 + 111x - 189$$

$$= x^3 - 21x^2 + 119x - 27$$

Then,

$$f(8) = 8^3 - 21 \times 8^2 + 119 \times 8 - 27$$

$$= 93.$$

Newton's Forward difference Interpolation formula:

$$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_n = x_0 + nh$$

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \frac{p(p-1)(p-2)(p-3)\Delta^4 y_0}{4!} + \dots$$

where,

$$p = \frac{x - x_0}{h} \quad \text{i.e. } x = x_0 + ph.$$

Formation of forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	Δ				
x_1	y_1	$\Delta y_0 = y_1 - y_0$				
x_2	y_2	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$			
x_3	y_3	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$		
x_4	y_4	$\Delta y_3 = y_4 - y_3$	$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$	
x_5	y_5	$\Delta y_4 = y_5 - y_4$	$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$	$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$	$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1$	$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0$

$$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0$$

Q. Find the cubic polynomial which takes following values:

x	0	1	2	3
$f(x)$	1	2	-1	10

Also find $f(4)$ and $f'(4)$

Sol.

The forward difference table for given data points,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 1$			
$x_1 = 1$	$y_1 = 2$	$\Delta y_0 = 1$		
$x_2 = 2$	$y_2 = -1$	$\Delta y_1 = -1$	$\Delta^2 y_0 = -2$	
$x_3 = 3$	$y_3 = 10$	$\Delta y_2 = 9$	$\Delta^2 y_1 = 10$	$\Delta^3 y_0 = 12$

here,

$$x_0 = 0, y_0 = 1, h = x_1 - x_0 = 1$$

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

now,

Newton's forward difference interpolation formula,

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

$$= 1 + x \cdot 1 + \frac{x(x-1)(-2)}{2} + \frac{x(x-1)(x-2) \cdot 1}{3 \cdot 2}$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1$$

$$f'(x) = 6x^2 - 14x + 6$$

Then,

$$f(4) = 2 \times 4^3 - 7 \times (4^2) + 6 \times (4) + 1$$

$$= 41$$

$$f'(4) = 6 \times 4^2 - 14 \times 4 + 6$$

$$= 46$$

Derivation of Newton's forward difference interpolation:-

→ Newton's form of polynomial is,

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \quad \text{--- (1)}$$

Let,

$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ be interpolating points. Also suppose the points are at equal distances

[equality means equal distance]

i.e.

$$x_0 + ih, \text{ where } i = 0, 1, 2, \dots, n.$$

If

$$x = x_0 \text{ then } P_n(x_0) = a_0 = b_0$$

If

$$x = x_1 \text{ then } P_n(x_1) = b_0 + a_1(x_1 - x_0) = b_1$$

$$a_1 = \frac{b_1 - b_0}{x_1 - x_0}$$

$$= \frac{\Delta b_0}{h}$$

If

$$x = x_2 \text{ then } P_n(x_2) =$$

$$\Rightarrow b_0 + \frac{b_1 - b_0}{x_1 - x_0} (x_2 - x_0) + a_2 (x_2 - x_0)(x_2 - x_1) = b_2$$

$$a_1 f_0 + \frac{f_1 - f_0}{h} \times 2h + a_2 \times 2h \times h = b_2$$

$$a_1 b_0 + (f_1 - f_0) \times 2 + 2h^2 a_2 = b_2$$

$$a_1 a_2 = \frac{b_2 - b_0 - 2f_1 + 2f_0}{2h^2}$$

$$a_1 a_2 = \frac{b_2 - 2f_1 + f_0}{2h^2}$$

$$a_2 = \frac{\Delta^2 f_0}{2h^2}$$

Putting the value of a_0 , a_1 and a_2 in eqⁿ (1),

$$P_n(x) = f_0 + \frac{\Delta f_0}{h} (x - x_0) + \frac{\Delta^2 f_0}{2h^2} (x - x_0)(x - x_1) + \dots$$

Let,

$$\text{index } S = p = \frac{x - x_0}{h}$$

$$P_n(x) = f_0 + p \Delta f_0 + \frac{\Delta^2 f_0}{2} p(p-1) + \dots$$

$$P_n(x) = f_0 + p \Delta f_0 + \frac{p(p-1) \Delta^2 f_0}{2!} + \frac{p(p-1)(p-2) \Delta^3 f_0}{3!} + \dots$$

This is called Newton's forward difference interpolation or this is also called Newton's Gregory forward interpolation formula.

Q. Construct Newton's forward-difference table for data points given in table below and then approximate the value of $f(1.1)$ by using Newton's forward difference formula and fourth divided difference.

x	1.0	1.3	1.6	1.9	2.2
$f(x)$	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

Sol.

The forward difference table for given data points,

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1.0	0.7651977				
1.3	0.6200860	-0.1451117			
1.6	0.4554022	-0.1646838	-0.0195721		
1.9	0.2818186	-0.1735836	-0.0088998	0.0106723	
2.2	0.1103623	-0.1714563	0.0021278	0.0110271	0.0003548

here,

$$x_0 = 1.0, y_0 = 0.7651977, h = x_1 - x_0 = 0.3$$

$$p = \frac{x - x_0}{h} = \frac{x - 1}{0.3}$$

Now,

Newton's forward difference interpolation formula;

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 0.7651977 + \left(\frac{x-1}{0.3} \right) (-0.01451117) +$$

$$\frac{\left(\frac{x-1}{0.3} \right) \left(\left(\frac{x-1}{0.3} \right) - 1 \right) (-0.0195721)}{2!} +$$

$$\frac{\left(\frac{x-1}{0.3} \right) \left(\left(\frac{x-1}{0.3} \right) - 1 \right) \left(\left(\frac{x-1}{0.3} \right) - 2 \right) \left(\frac{x-1}{0.3} \right) (0.0106723)}{3!} +$$

$$\frac{\left(\frac{x-1}{0.3} \right) \left(\left(\frac{x-1}{0.3} \right) - 1 \right) \left(\left(\frac{x-1}{0.3} \right) - 2 \right) \left(\left(\frac{x-1}{0.3} \right) - 3 \right) (0.0003548)}{4!}$$

$$\underline{P(1.1) =}$$

$$0.7651977 + (-0.04837056667) + \\ (-2.174677778 \times 10^{-3}) + (6.587839506 \times 10^{-4}) \\ + (-1.660882305 \times 10^{-5})$$

$$= 0.721820672.$$

Newton's Backward difference interpolation:

Newton's form of polynomial is,

$$P_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \dots + a_n(x-x_n)(x-x_{n-1})\dots(x-x_1) \quad \text{--- (1)}$$

our objective to find $a_0, a_1, a_2, \dots$

if $x = x_n$ then $P_n(x_n) = a_0 = f_n$

if $x = x_{n-1}$ then $P_n(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n) = f_{n-1}$

or, $f_n + a_1(x_{n-1} - x_n) = f_{n-1}$

or, $f_n - f_{n-1} = -a_1(x_{n-1} - x_n)$

$$\therefore a_1 = \frac{f_n - f_{n-1}}{x_n - x_{n-1}}$$

$$\therefore a_1 = \frac{\nabla f_n}{h}$$

if $x = x_{n-2}$ then

$$P_n(x_{n-2}) = f_n + \frac{f_n - f_{n-1}}{x_n - x_{n-1}}(x_{n-2} - x_n) +$$

$$a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) = f_{n-2}$$

or, $f_n + \frac{f_n - f_{n-1}}{h}(-2h) + a_2(-2h)(-h) = f_{n-2}$

$$a_1, \quad f_n + (f_n - f_{n-1})(-2) + a_2(2h^2) = f_{n-2}$$

$$a_1, \quad f_n + 2f_{n-1} - 2f_n + 2h^2 a_2 = f_{n-2}$$

$$a_1, \quad 2h^2 a_2 = f_{n-2} - f_n + 2f_{n-1}$$

$$a_1, \quad a_2 = \frac{f_{n-2} - f_n + 2f_{n-1}}{2h^2}$$

$$a_1, \quad a_2 = \frac{f_{n-2} + 2f_{n-1} - f_n}{2h^2}$$

$$a_1, \quad a_2 = \frac{f_n - 2f_{n-1} + f_{n-2}}{2h^2}$$

$$\boxed{a_2 = \frac{\nabla^2 f_n}{2h^2}}$$

putting the value of a_0, a_1, a_2 in eqⁿ ①, we get

$$P_n(x) = f_n + \frac{\nabla f_n}{h}(x - x_n) + \frac{\nabla^2 f_n}{2h^2}(x - x_n)(x - x_{n-1}) + \dots$$

$$\text{Let index, } s = p = \frac{x - x_n}{h}$$

$$P_n(x) = f_n + p \nabla f_n + \frac{p(p+1)}{2!} \nabla^2 f_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 f_n + \dots$$

This is called Newton's Gregory backward difference interpolation formula.

Formation of N-B-D-I-Formula Table:

x	y	∇y	$\nabla^2 y$
x_0	y_0		
x_1	y_1	$\nabla y_1 = y_1 - y_0$	
x_2	y_2	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$
x_3	y_3	$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$
x_4	y_4	$\nabla y_4 = y_4 - y_3$	$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$

$$\nabla^3 y$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$$

$$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$$

$$\nabla^4 y$$

$$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$$

Q Using N-B-I-F find $f(42)$ for the following data.

x	20	25	30	35	40	45
$f(x)$	354	332	292	260	231	204

solⁿ:

The backward difference interpolation table for given data points are,

x	$f(x)=y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	354					
25	332	-22				
30	292	-41	-19			
35	260	-32	10	29		
40	231	-29	2	-8	-37	
45	204	-27	2	0	8	45

$$x_n = 45, \quad x = 42, \quad h = x_1 - x_0 = 25 - 20 = 5$$

$$p = \frac{x - x_n}{h} = \frac{42 - 45}{5} = -\frac{3}{5} = -0.6$$

$$y_n = 204$$

now,

Newton's backward interpolation formula is,

$$P_n(x) = y_n + p \Delta y_n + \frac{p(p+1)}{2!} \Delta^2 y_n + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!} \Delta^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \Delta^5 y_n$$

$$= 204 + (-0.6+1) \Delta y_n + \frac{(-0.6)(-0.6+1) \times 2}{2!} \Delta^2 y_n + \frac{(-0.6)(-0.6+1)(-0.6+2) \times 0}{3!} + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!} \times 8 + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4) \times 45}{5!}$$

$$= 204 + (-10.8) + (-0.24) + 0 + (-0.2688) + (-1.02816)$$

$$= 191.66304 //$$

Regression.

- Regression is a way of fitting the curve or straight line through the given data points.
- Investigate relationship between dependent variable (target) and independent variable (predictors).
- Here we fit a curve or line to the data points, in such a manner that the difference between the distance of data points from the curve or line is minimized.
- Let the set of data points $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$ and let the curve given by $y = f(x)$ be fitted on this data. At $x = x_i$, the experimental (or observed) value of the ordinate is y_i and corresponding value on the fitted curve is $f(x_i)$ if e_i is the error of approximation at $x = x_i$, then we have
$$e_i = y_i - f(x_i) \text{ --- (1)}$$

Let us denote :

$$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \dots + [y_n - f(x_n)]^2$$
$$= e_1^2 + e_2^2 + \dots + e_n^2 \text{ --- (2)}$$

Then this method of least square consists of minimizing S .
i.e. the sum of squares of errors.

Linear Regression (Fitting a straight line):

→ Let $y = f(x) = ax + b$ be the straight line to be fitted to the given data, where 'a' is the slope and b is y-intercept. (a and b are regression coefficient).

→ Let the Sum of squares of individual errors can be expressed as:

$$S = \sum e_i^2 = \sum (y_i - f(x_i))^2$$

$$= \sum_{i=1}^n (y_i - ax_i - b)^2 \quad \text{--- (3)}$$

$$S = [y_1 - (ax_1 + b)]^2 + [y_2 - (ax_2 + b)]^2 + \dots + [y_n - (ax_n + b)]^2$$

→ In the least square method, we choose 'a' and 'b' such that S is minimum.

Since,

'S' depends on 'a' and 'b' necessary condition for S to be minimum is,

$$\frac{\partial S}{\partial a} = 0 \quad \text{and} \quad \frac{\partial S}{\partial b} = 0$$

Now,

$$\frac{\partial S}{\partial a} = \sum_{i=1}^n 2 (y_i - ax_i - b) (-x_i)$$

$$0 = -2 \sum_{i=1}^n (x_i^0) (y_i^0 - ax_i^0 - b)$$

$$\therefore \sum_{i=1}^n (x_i^0) (y_i^0 - ax_i^0 - b) = 0 \quad \text{--- (4)}$$

Simplifying

$$\frac{\partial S}{\partial b} = 0$$

i.e.

$$\sum_{i=1}^n 2 (y_i^0 - ax_i^0 - b) (-1) = 0$$

$$\therefore \sum_{i=1}^n (y_i^0 - ax_i^0 - b) = 0 \quad \text{--- (5)}$$

From equation (4), we get

$$\sum_{i=1}^n (x_i^0) (y_i^0 - ax_i^0 - b) = 0$$

$$\therefore \sum x_i^0 y_i^0 - a \sum x_i^0{}^2 - b \sum x_i^0 = 0$$

$$\therefore \sum x_i^0 y_i^0 = a \sum x_i^0{}^2 + b \sum x_i^0 \quad \text{--- (6)}$$

From equation (5), we get

$$\sum_{i=1}^n (y_i^0 - ax_i^0 - b) = 0$$

$$\sum y_i - a \sum x_i - \sum b = 0$$

$$\sum y_i = a \sum x_i + bn$$

$$\text{or, } bn = \sum y_i - a \sum x_i$$

$$\therefore b = \frac{\sum y_i - a \sum x_i}{n} \quad \text{--- (6)}$$

$$\text{or, } b = \bar{y} - a\bar{x} \quad \text{--- (7)}$$

putting the value of b in eqⁿ (6), we get

$$\sum x_i y_i = a \sum x_i^2 + b \sum x_i$$

$$\text{or, } \sum x_i y_i = a \sum x_i^2 + \left(\frac{\sum y_i - a \sum x_i}{n} \right) \sum x_i$$

$$\text{or, } n \sum x_i y_i = na \sum x_i^2 + \sum x_i \sum y_i - a (\sum x_i)^2$$

$$\text{or, } n \sum x_i y_i - \sum x_i \sum y_i = a (n \sum x_i^2 - (\sum x_i)^2)$$

$$\therefore a = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

example :

Use least square method to fit a straight line to the following data.

x	1	3	4	6	8	9	11
y	1	2	4	4	5	7	8

Sol :

Let, straight line eqⁿ is $y = ax + b$ — (1)

$$a = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$b = \bar{y} - a \bar{x}$$

$$\bar{y} = \frac{\sum y_i}{n}, \quad \bar{x} = \frac{\sum x_i}{n}$$

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
1	1	1	1	1
3	2	9	4	6
4	4	16	16	16
6	4	36	16	24
8	5	64	25	40
9	7	81	49	63
11	8	121	64	88

$$\sum x_i = 42 \quad \sum y_i = 31 \quad \sum x_i^2 = 328 \quad \sum y_i^2 = 175 \quad \sum x_i y_i = 238$$

$$a = \frac{7 \times 238 - 42 \times 31}{7 \times 828 - (42)^2}$$

$$= 0.6842$$

$$b = \bar{y} - a\bar{x}$$

$$\therefore \bar{y} = \frac{\sum y_i}{n}$$

$$= \frac{31}{7}$$

$$= 4.4285$$

$$\bar{x} = \frac{\sum x_i}{n}$$

$$= \frac{42}{7}$$

$$= 6$$

$$\begin{aligned} b &= 4.4285 - 0.6842 \times 6 \\ &= 4.4285 - 4.1052 \\ &= 0.3233 \end{aligned}$$

From eqⁿ (1),

$$y = 0.6842x + 0.3233$$

Q. The values of x and y are shown in table:

x	0	1	2	3	4
y	2	3	5	4	6

- i) Find the least square regression line.
 ii) Estimate the value of y for given $x=10$.

Sol:

x	y	x^2	y^2	xy
0	2	0	4	0
1	3	1	9	3
2	5	4	25	10
3	4	9	16	12
4	6	16	36	24
$\Sigma x = 10$	$\Sigma y = 20$	$\Sigma x^2 = 30$	$\Sigma y^2 = 90$	$\Sigma xy = 49$

The regression line is,

$$y = ax + b \quad \text{--- (1)}$$

$$a = \frac{n \Sigma xy - \Sigma x \Sigma y}{n \Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{5 \times 49 - 10 \times 20}{5 \times 30 - (10)^2}$$

$$= 0.9$$

$$b = \bar{y} - a\bar{x}$$

$$= \frac{\sum y}{n} - a \times \frac{\sum x}{n}$$

$$= \frac{20}{5} - 0.9 \times \frac{10}{5}$$

$$= 4 - 0.9 \times 2$$

$$= 2.2$$

from eqⁿ (1),

$$(i) \quad y = 0.9x + 2.2$$

is the required estimated regression equation.

(ii) for $x = 10$,

$$y = 0.9 \times 10 + 2.2$$

$$= 9 + 2.2$$

$$= 11.2$$

//

Fitting Exponential model:-

Given $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$, best fit $y = ae^{bx}$. The variables a and b are constants of the exponential model. The residual at each data point x_i is error.

$$e_i = y_i - ae^{bx_i} \quad \text{--- (1)}$$

The sum of the square of residual is,

$$E = \sum_{i=1}^n e_i^2$$

$$E = \sum_{i=1}^n (y_i - ae^{bx_i})^2 \quad \text{--- (2)}$$

To find the constants a and b we minimize E by differentiating with respect to a & b and equating the resulting eq's to zero.

$$\frac{\partial E}{\partial a} = \sum_{i=1}^n 2(y_i - ae^{bx_i})(-e^{bx_i}) = 0 \quad \text{--- (3)}$$

$$\frac{\partial E}{\partial a} = \sum_{i=1}^n 2(y_i - ae^{bx_i})(-ae^{bx_i}) = 0 \quad \text{--- (4)}$$

From eq (3), we get

$$\frac{\partial E}{\partial a} = \sum_{i=1}^n 2(y_i - ae^{bx_i}) \cdot (-e^{bx_i}) = 0$$

$$\Rightarrow - \sum_{i=1}^n y_i e^{bx_i} + a \sum_{i=1}^n e^{2bx_i} = 0$$

$$\therefore a = \frac{\sum_{i=1}^n y_i e^{bx_i}}{\sum_{i=1}^n e^{2bx_i}}$$

From eqⁿ (4), we get

$$\frac{\partial E}{\partial b} = \sum_{i=1}^n 2 (y_i - a e^{bx_i}) (-a x_i e^{bx_i}) = 0$$

$$\sum_{i=1}^n (-y_i \cdot a x_i e^{bx_i}) + \sum_{i=1}^n a \cdot a x_i e^{2bx_i} = 0$$

$$a \left\{ \sum_{i=1}^n (-y_i x_i e^{bx_i}) + \sum_{i=1}^n a x_i e^{2bx_i} \right\} = 0$$

$$\sum_{i=1}^n y_i x_i e^{bx_i} - a \sum_{i=1}^n x_i e^{2bx_i} = 0 \quad \text{--- (5)}$$

Substituting the value of a in eq⁵, we get

$$\sum_{i=1}^n y_i x_i e^{bx_i} = \frac{\sum_{i=1}^n y_i e^{bx_i}}{\sum_{i=1}^n e^{2bx_i}} \times \sum_{i=1}^n x_i e^{2bx_i}$$

The eqⁿ is still a non-linear eqⁿ in b and can be solved by Numerical Method such as bisection or secant.

Fitting exponential model by Linearization.

Exponential model is given by,

$$y = ae^{bx} \text{ --- (1)}$$

Taking natural log on both sides,

$$\log y = \log a + \log e^{bx}$$

$$\log y = \log a + bx \text{ --- (2)}$$

This eqⁿ is similar to the form of linear equation $y = a + bx$

Thus we can evaluate parameters a and b by using the eqⁿ of linear regression model.

$$b = \frac{n \sum_{i=1}^n x_i \log y_i - \sum_{i=1}^n x_i \sum_{i=1}^n \log y_i}{n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2} \quad (3)$$

$$\log a = \frac{\sum_{i=1}^n \log y_i}{n} - b \frac{\sum_{i=1}^n x_i}{n} \quad (4)$$

$$\log a = \frac{\sum_{i=1}^n \log y_i}{n} - b \frac{\sum_{i=1}^n x_i}{n}$$

Let,

$$R = \frac{\sum_{i=1}^n \log y_i}{n} - b \frac{\sum_{i=1}^n x_i}{n}$$

Now,

$$\text{we have } \log a = R$$

Taking antilog on both sides, we get

$$a = e^R \text{ --- (5)}$$

Now, we can evaluate the value of regression coefficient a and b .

Q. Fit the curve $y = ae^{bx}$ through the data given below:

X	-4	-2	0	1	2	4
Y	0.57	1.32	4.12	6.65	11	30.3

Sol:

we know that the exponential model is,
 $y = ae^{bx}$

$$\Rightarrow \log y = \log a + bx$$

$$b = \frac{n \sum x_i \log y_i - \sum x_i \sum \log y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\log a = \frac{\sum \log y_i}{n} - b \frac{\sum x_i}{n}$$

x_i	y_i	$\log y_i$	$x_i \log y_i$	x_i^2
-4	0.57	-0.5622	-2.2484	16
-2	1.32	0.2776	-0.5552	4
0	4.12	2.4158	0	0
1	6.65	2.8946	2.8946	1
2	11	2.39789	4.7957	4
4	30.3	3.4111	13.6445	16

$$\sum x_i = 1$$

$$\sum y_i = 53.96$$

$$\sum \log y_i = 8.8348$$

$$\sum x_i \log y_i = 22.03$$

$$\sum x_i^2 = 41$$

$$b = \frac{6 \sum x_i \log y_i - \sum x_i \sum \log y_i}{6 \sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{6 \times 22.03 - 1 \times 8.8348}{6 \times 41 - (1)^2}$$

$$= 0.503449$$

$$\log a = \frac{\sum \log y_i}{6} - \frac{b \sum x_i}{6}$$

$$= \frac{8.8348}{6} - \frac{(0.503449) \times 1}{6}$$

$$\log a = 1.38855$$

$$a = e$$

$$a = 4.009$$

Then,

$$y = 4.009 \times e^{0.503449x}$$

Q Below is given the relative intensity of radiation as a function of time.

x (time)	0	1	3	5	7	9
y = intensity of radiation	1	0.892	0.708	0.562	0.447	0.355

If the level of the relative intensity of radiation is related to time via an exponential formula $y = ae^{bx}$, find the value of the regression constants a and b . Also estimate intensity of radiation at time $t = 13$.

Sol:

x_i	y_i	$\ln y_i$	$x_i \ln y_i$	x_i^2
0	1	0	0	0
1	0.892	-0.1154	-0.1154	1
3	0.708	-0.3453	-1.03593	9
5	0.562	-0.5762	-2.8812	25
7	0.447	-0.80519	-5.63637	49
9	0.355	-1.0356	-9.3207	81

$$\sum x_i = 25 \quad \sum y_i = 3.963 \quad \sum \ln y_i = -2.877 \quad \sum x_i \ln y_i = -18.98 \quad \sum x_i^2 = 165$$

$$b = \frac{n \sum x_i \log y_i - \sum x_i \sum \log y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{6 \times (-18.98) - 25 \times (-2.877)}{6 \times 165 - (25)^2}$$

$$= -0.115049$$

$$\log a = \frac{\sum \ln y_i}{n} - \frac{b \sum x_i}{n}$$

$$= \frac{-2.87}{6} - \frac{(-0.115049) \times 25}{6}$$

$$\log a = -0.000261498$$

$$a = e^{-0.000261498}$$

$$a = 0.999738$$

Then,

$$y = a e^{bx}$$

$$= 0.999738 \times e^{(-0.115049) \times 13}$$

$$= 0.224045$$

Numerical Differentiation and integration.

→ The derivative of differentiation of a function represents the rate of change of variable with respect to another variable.

→ Numerical differentiation is the process of obtaining the value of the derivative of a function from a set of numerical values of that values.

→ There are basically two situations where numerical differentiation is required.

i) The function values are known but the function is unknown, such functions are called tabulated function.

ii) The functions to be differentiated is complicated and it is difficult to differentiate.

Differentiating Continuous function!

To point forward difference formula,

According to Taylor's theorem
if we know the values of a function
 $f(x)$ at point x_0 and all its derivatives
at that point provided that derivatives
are continuous between x_0 and $x_0 + 1$

$$f(x_0 + 1) = f(x_0) + f'(x_0)$$

For convenience,

$$h = x_0 + 1 - x_0$$

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2!} + \dots$$

$$\therefore f(x_0 + h) - f(x_0) = \frac{f''(x_0)h^2}{2!} + \dots = f'(x_0)h$$

$$\therefore f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{f''(x_0)h}{2!} + \dots$$

$$\therefore f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + E \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\therefore \boxed{f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h}} \quad \text{--- (1)}$$

Here, E is the error term in the approximation. This eqn (1) is called two point forward difference formula.

Algorithm

- 1) Start
- 2) Read the value at which derivative is needed, say x .
- 3) Read interval gap, say h .
- 4) Calculate $f(x_0)$ and $f(x_0+h)$.
- 5) Calculate

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

- 6) Display the result.
- 7) End

Q Find the value of derivative at $x=1$, for the function $f(x) = x^2$.
Use $h=0.2, 0.05$.

Solⁿ,

For $h=0.2$

We know,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(1) = \frac{f(1+0.2) - f(1)}{0.2}$$

$$f'(1) = \frac{f(1.2) - f(1)}{0.2}$$

$$= \frac{1.44 - 1}{0.2}$$

$$f'(1) = 2.2$$

for $h=0.05$

we know,

$$f'(1) = \frac{f(1+0.05) - f(1)}{0.05}$$

$$f'(1) = 2.05$$

9 Find the value of derivative at $x=45^\circ$ for the function $f(x) = \sin x + 1$ by using $h=0.1$ and 0.01 .

(Degree to Radian)
 $\left(\frac{\pi}{180} \times x \right)$

Solⁿ:

For $h=0.1$,

$$x = 45^\circ$$

change the 45° to gradient,

$$x = \frac{\pi}{180} \times 45$$

$$= \pi/4$$

For $h=0.1$,

we know,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(\pi/4) = \frac{f(\pi/4+0.1) - f(\pi/4)}{0.1}$$

$$= \frac{1.0154 - 1.013707}{0.1}$$

$$= 0.01745 //$$

For $h=0.01$,

we know,

$$f'(\pi/4) = \frac{f(\pi/4+0.01) - f(\pi/4)}{0.01}$$

$$= \frac{1.0154 - 1.0137}{0.01}$$

$$= 0.174514 //$$

Two point Backward difference formula:

→ Taylor Theorem.

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

Here,

x_{i+1} is behind x_i , so for convenience $h = x_i - x_{i+1}$

$$f'(x_i - h) = f(x_i) - f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \dots$$

$$\therefore f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h} = \frac{f''(x_i)h}{2!} + \dots$$

$$\therefore f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h} + E$$

$$\approx \frac{f(x_i) - f(x_i - h)}{h}$$

$$\boxed{f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h}}$$

Here E is the error term in approximation.
This eqⁿ is called two point backward difference formula.

Q Find the value of the derivative at $x=1$ for the function $f(x)=x^2$ by using $h=0.2$ and $h=0.05$.

Sol:

for $h=0.2$,

$$f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h}$$

$$f'(1) = \frac{f(1) - f(1-0.2)}{0.2}$$

$$= \frac{1 - 0.64}{0.2}$$

$$= 1.8$$

for $h=0.05$,

$$f'(1) = \frac{f(1) - f(1-0.05)}{0.05}$$

$$= 1.95$$

//

* Find the value of derivate at $x = 45^\circ$ for the function $f(x) = \sin x + 1$ by using two point backward difference formula, $h = 0.1, 0.01$

Sol:

$$x = 45^\circ$$

$$x = \frac{\pi}{180} \times 45$$

$$x = \pi/4$$

for $h = 0.1$

$$f'(\pi/4) = \frac{f(\pi/4) - f(\pi/4 - 0.1)}{0.1}$$

$$= 0.01745$$

for $h = 0.01$,

$$f'(\pi/4) = \frac{f(\pi/4) - f(\pi/4 - 0.01)}{0.01}$$

$$= 0.01745$$

Three points formula : (it is also called central difference)

→ Taylor series :

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2!} + \dots \quad (1)$$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)h^2}{2!} + \dots \quad (2)$$

Subtracting eqⁿ (2) from eqⁿ (1), we get,

$$f(x_0+h) - f(x_0-h) = f'(x_0)2h + \dots$$

$$\therefore f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + E$$

$$\approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

i.e.

$$\boxed{f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}} \quad (3)$$

This eqⁿ is called three points formula.

Find the value of derivative at $x=1$ for the function $f(x)=x^2$ by using $h=0.2$ and 0.05 from three points formula.

solⁿ →

for $h=0.2$

$$f'(1) = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2}$$

$$= 2$$

for $h=0.05$,

$$f'(1) = \frac{f(1+0.05) - f(1-0.05)}{2 \times 0.05}$$

$$= 2$$

Q Find the value of derivate at $x=45^\circ$ for the function of $f(x) = \sin x + 2$ by using three point backward difference formula at $h=0.1$ and $h=0.01$.

Solⁿ:

$$x = 45^\circ$$

$$x = \frac{\pi}{180} \times 45$$

$$x = \pi/4$$

For $h = 0.1$,

$$f'(\pi/4) = \frac{f(\pi/4 + 0.1) - f(\pi/4 - 0.1)}{2 \times 0.1}$$

$$= 0.0174516$$

//

for $h = 0.01$,

$$f'(\pi/4) = \frac{f(\pi/4 + 0.01) - f(\pi/4 - 0.01)}{2 \times 0.01}$$

$$= 0.0174516$$

//

Differentiating Discrete function :

Derivating using Newton's Divide difference :

Formula :

Newton's form of polynomial is :

$$P_n(x) = f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$\text{or, } P_n(x) = f(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \dots + f[x_0, x_1, x_2, \dots, x_n](x-x_0)(x-x_1)\dots(x-x_{n-1}) \quad \text{--- (1)}$$

$$P_n(x) = f[x_0] + \sum_{j=1}^n f[x_0, x_1, \dots, x_n] \prod_{j=0}^{n-1} (x-x_j)$$

Differentiating (1) w.r.t. x we get,

$$f'(x) = f[x_0, x_1] + f[x_0, x_1, x_2] \{ (x-x_1) + (x-x_0) \} + f[x_0, x_1, x_2, x_3] \{ (x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1) \} \quad \text{--- (2)}$$

By putting $x = a$ in eqⁿ (2) we get the value of 1st derivative at $x = a$.

Q A slider in a machine moves along a fixed straight rod. Find the velocity of slider when $t = 0.3$ seconds.

Time (x)	0.1	0.2	0.3	0.4	0.5
Distance $f(x)$	31.62	32.87	33.64	33.95	33.82

~~901~~

	x	$f(x)$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$
x_0	0.1	31.62	$\frac{32.87 - 31.62}{0.2 - 0.1} = 12.5$	$\frac{33.64 - 12.5}{0.3 - 0.2} = -23$
x_1	0.1 0.2	32.87		
x_2	0.3	33.64	7.7	-23
x_3	0.4	33.95	3.1	
x_4	0.5	33.82	-1.3	-22

$f[x_0, x_1, x_2, x_3]$

$f[x_0, x_1, x_2, x_3, x_4]$

3.33

0

3.33

$$f'(x) = f[x_0, x_1] + f[x_0, x_1, x_2] \{ (x - x_1) + (x - x_0) \} + f[x_0, x_1, x_2, x_3] \{ (x - x_1)(x - x_2) + (x - x_0)(x - x_2) + (x - x_0)(x - x_1) \}$$

$$= 12.5 + (-24) \{ (0.3 - 0.2) + (0.3 - 0.1) \} + (3.33) \{ (0.3 - 0.2)(0.3 - 0.3) + (0.3 - 0.1)(0.3 - 0.3) + (0.3 - 0.1)(0.3 - 0.2) \}$$

$$= 5.3666$$

//

Derivatives using Newton's forward difference formula

2) $(x_0, f(x_0)), (x_1, f(x_1)) \dots (x_n, f(x_n))$ can be written as,

$$P_n(x) = f_0 + p \Delta f_0 + \frac{p(p-1)}{2!} \Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 f_0 + \dots \quad (1)$$

where,

$$p = \frac{x - x_0}{h}$$

$$\therefore h dp = dx$$

Differentiating eqⁿ (1) w.r.t. x we get

$$P'_n(x) = \frac{d[f(x)]}{dx} \Rightarrow \frac{d[P(x)]}{h dp}$$

$$= \frac{1}{h} \left[\Delta f_0 + \frac{1}{2!} (2p-1) \Delta^2 f_0 + \frac{1}{3!} (3p^2 - 6p + 2) \Delta^3 f_0 + \right.$$

$$\left. \frac{1}{4!} (4p^3 - 18p^2 + 22p - 6) \Delta^4 f_0 + \dots \right] \quad (2)$$

By putting $x = a$, eqⁿ (2) can be used to find the value of the first derivative at $x = a$.

Again differentiating eq (2) w.r.t. x , we get

$$p''(x) = \frac{1}{h^2} \left[\Delta^2 f_0 + \frac{1}{3!} (6p-6) \Delta^3 f_0 + \frac{1}{4!} (12p^2-36p+22) \Delta^4 f_0 \right] \quad (3)$$

By putting $x=a$, eq (3) can be used to find the value of 2nd derivative at point $x=a$.

Q Find the first and second derivative of the function tabulated below at 1.1.

x	1.0	1.2	1.4	1.6	1.8	2
$f(x)$	0	0.128	0.544	1.296	2.432	4

Sol

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1.0	0					
1.2	0.128	0.128	0.288			
1.4	0.544	0.416	0.336	0.048	0	
1.6	1.296	0.752	0.384	0.048	0	0
1.8	2.432	1.136	0.432	0.048	0	
2.0	4	1.568				

$$h = 1.2 - 1 = 0.2$$

$$p = \frac{x - x_0}{h} = \frac{1.1 - 1}{0.2} = 0.5$$

$$P_n'(x) = \frac{1}{h} \left[\Delta f_0 + \frac{1}{2!} (2p-1) \Delta^2 f_0 + \frac{1}{3!} (3p^2-6p+2) \Delta^3 f_0 \right]$$

$$= \frac{1}{0.2} \left[0.128 + \frac{1}{2!} (2 \times 0.5 - 1) \times 0.288 + \frac{1}{3!} (3 \times 0.5^2 - 6 \times 0.5 + 2) 0.048 \right]$$

$$= 0.63$$

$$P_n''(x) = \frac{1}{h} \left[\Delta^2 f_0 + \frac{1}{3!} (6p-6) \Delta^3 f_0 \right]$$

$$= \frac{1}{0.2} \left[0.288 + \frac{1}{3!} (6 \times 0.5 - 6) 0.048 \right]$$

$$= 1.056$$

Numerical Integration:

In general problem of numerical integration is stated as follows:

Given a set of data points $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$ of a function the value $y = f(x)$, where, $f(x)$ is not known explicitly, it is required to compute the values of definite integral.

$$I = \int_a^b y dx.$$

Here, we replace $f(x)$ by an interpolating function $p(x)$ or $q(x)$ and on integration obtain an approximation value of the definite integral.

Depending on the type of $p(x)$ or $q(x)$ is used, different integral formula can be obtained.

Graphical Representation of a definite integral:

The definite integral $I = \int_a^b f(x) dx$ can be treated as the area under the curve $y = f(x)$ enclosed between the limit $x = a$ & $x = b$, as shown in figure -:

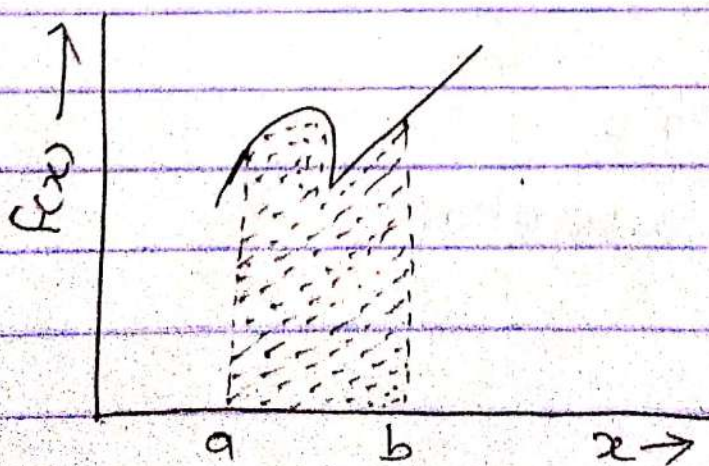


fig: Graphical Representation of integral of a function

The problem of integration is then reduced to the problem of finding the shaded area.

General formula for numerical integration!
(Using Newton's forward difference formula);

→ Let the interval $[a, b]$ be divided into n equal subintervals such that,

$$a = x_0 < x_1 < x_2 < \dots < x_n = b \text{ clearly}$$
$$x_n = x_0 + nh.$$

Hence, integration becomes:

$$I = \int_{x_0}^{x_n} y dx$$

Approximating 'y' by Newton's forward difference formula:

$$I = \int_{x_0}^{x_n} \left[y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \right] dx$$

Since,

$$x = x_0 + ph$$

$$\frac{dx}{dp} = h \Rightarrow dx = h dp$$

$$I = h^p \left[y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \right] dp$$

On simplification, we get

$$\int_{x_0}^x y dx = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right] \quad (1)$$

which is the general formula. we obtain different formula by putting $n=1, 2, 3, \dots$

Trapezoidal Rule :

Setting $n=2$ in general formula for integration all differences higher than 1 (1st) will become 0 and we obtain,

$$\int_{x_0}^{x_2} y dx = nh \left[y_0 + \frac{1}{2} \Delta y_0 \right]$$

$$= 2h \left[y_0 + \frac{y_1 - y_0}{2} \right] \quad \because \Delta y = [y_1 - y_0]$$

$$= h \left[\frac{y_0 + y_1}{2} \right]$$

$$= \frac{h}{2} [y_0 + y_1]$$

Similarly,

$$\int_{x_1}^{x_2} y dx = h/2 [y_1 + y_2]$$

$$\int_{x_2}^{x_3} y dx = h/2 [y_2 + y_3]$$

$$\vdots$$

$$\int_{x_{n-1}}^{x_n} y dx = h/2 [y_{n-1} + y_n]$$

Combining all the expressions:

$$\int_{x_0}^{x_n} y dx = h/2 [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n]$$

Q Find $\int_2^8 (x^3+2)dx$ by using trapezoidal rule.

$$f(x_i) = x_i + ih$$

Sol.

$$x_0 = 2, x_1 = 8$$

$$\begin{aligned} f(x_0) &= f(x_0 + 0 \cdot h) & \therefore i=0 & \therefore h = x_1 - x_0 \\ &= f(2 + 0 \cdot 6) \\ &= f(2) = 2^3 + 2 = 10 \end{aligned}$$

$$\begin{aligned} f(x_1) &= f(x_0 + 1 \cdot h) \\ &= f(2 + 1 \cdot 6) \\ &= f(8) = 8^3 + 2 = 514 \end{aligned}$$

$$\begin{aligned} \int_{x_0}^{x_1} f(x) dx &= h \cdot \frac{1}{2} [f(x_0) + f(x_1)] \\ &= 6 \cdot \frac{1}{2} [10 + 514] \end{aligned}$$

$$= 1572 \quad //$$

Q Find $\int_0^1 e^{-x^2} dx$ by using trapezoidal rule.

Sol

$$x_0 = 0, x_1 = 1$$

$$f(x_0) = f(x_0 + 0 \cdot h)$$

$$= f(0 + 0)$$

$$= f(0)$$

$$= e^{-0^2}$$

$$= e^0$$

$$= 1$$

$$f(x_1) = f(x_0 + 1 \cdot h)$$

$$= f(0 + 1 \cdot 1)$$

$$= f(1)$$

$$= e^{-1^2}$$

$$= e^{-1}$$

$$= e^{-1}$$

$$= 0.3678$$

$$\int_{x_0}^{x_1} f(x) dx = h_{1/2} [f(x_0) + f(x_1)]$$

$$= \frac{1}{2} [0.8678 + 0.3678]$$

$$= 0.6839 //$$

Q Find the area bounded by the curve and the axis from $x=7.47$ to $x=7.52$.

x	$f(x)$
7.47	1.93
7.48	1.95
7.49	1.98
7.50	2.01
7.51	2.03
7.52	2.06

From the Trapezoidal rule, Area = $\int_{7.47}^{7.52} f(x) dx$

$$\text{Area} = h_{1/2} [y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5]$$

$$= \frac{0.01}{2} [1.93 + 2 * (1.95 + 1.98 + 2.01 + 2.03) + 2.06]$$

$$= 0.09965 //$$

Error calculation for the trapezoidal Rule

Let $y = f(x)$ be a continuous and possess continuous derivative in $[x_0, x_1]$ expanding y in a Taylor series around $x = x_0$.

$$\int_{x_0}^x y \, dx = \int_{x_0}^x \left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \dots \right] dx$$

$$= y_0 \left[x \right]_{x_0}^x + \left[\frac{(x-x_0)^2}{2} \right]_{x_0}^x y_0' + \left[\frac{(x-x_0)^3}{2^3} \right]_{x_0}^x y_0'' + \dots$$

$$= y_0 [x_1 - x_0] + \left[\frac{x_1 - x_0}{2} \right] y_0' + \frac{(x_1 - x_0)^3}{6} y_0''$$

$$\int_{x_0}^{x_1} y \, dx = h y_0 + \frac{h^2}{2} y_0' + \frac{h^3}{6} y_0'' + \dots \quad (*)$$

Similarly,

$$h_{1/2} [y_0 + y_1] = h_{1/2} \left[y_0 + y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \dots \right]$$

$$= h_{1/2} \left[2y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \dots \right]$$

$$= h y_0 + \frac{h^2}{2} y_0' + \frac{h^3}{4} y_0'' + \frac{h^4}{12} y_0''' + \dots \quad (**)$$

~~Sub~~

subtracting $(*)$ from $(*)$

$$\int_{x_0}^{x_1} y \, dx = \frac{h^3}{12} y_0'' + \frac{h^4}{24} y_0''' + \dots$$
$$= \frac{2h^3 y_0'' + h^4 y_0'''}{24}$$

which is the error in interval $[x_0, x_1]$ preceding
In similar means we obtain the error in
remaining sub intervals

$$[x_1, x_2], [x_2, x_3] \dots [x_{n-1}, x_n]$$

Simpson's $\frac{1}{3}$ Rule :

This rule is obtained by putting $n=2$ in general formula..

$$\int_{x_0}^{x_2} y dx = nh \left[y_0 + \frac{1}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^3 y_0 + \dots \right]$$

at $n=2$

$$= 2h \left[y_0 + \frac{1}{2} \Delta y_0 + \frac{1}{6} \Delta^2 y_0 + \dots \right]$$

$$= 2h \left[y_0 + y_1 - y_0 + \frac{1}{6} (y_2 - 2y_1 + y_0) \right]$$

$$= 2h \left[\frac{6y_1 + y_2 - 2y_1 + y_0}{6} \right]$$

$$= \frac{h}{3} [4y_1 + y_2 + y_0]$$

$$= \frac{h}{3} [y_0 + 4y_1 + y_2]$$

Similarly,

$$\int_{x_2}^{x_4} y dx = \frac{h}{3} [y_2 + 4y_3 + y_4]$$

$$\int_{x_4}^{x_6} y dx = \frac{h}{3} [y_4 + 4y_5 + y_6]$$

$$\int_{x_{n-2}}^{x_n} = h/3 [y_{n-2} + 4y_{n-1} + y_n]$$

Combining all expression.

$$\int_{x_0}^{x_n} = h/3 [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + y_n]$$

$$= h/3 [y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2}) + y_n]$$

which is known as Simpson's $1/3$ Rule.

Q Apply Simpson's $\frac{1}{3}$ Rule to calculate $\int_0^1 \sqrt{1-x^2} dx$

$$h = \frac{1-0}{2} = 0.5$$

$$f(x) = \sqrt{1-x^2}, \quad x_0 = 0, \quad x = 1$$

$$f(x_0) = f(x_0 + 0h) = f(0 + 0 \times h) = f(0) = 1$$

$$f(x_1) = f(x_0 + 1h) = f(0 + 0.5) = f(0.5) = 0.866$$

$$f(x_2) = f(x_0 + 2h) = f(0 + 2 \times 0.5) = f(1) = 0$$

Now,

$$\int_0^1 y dx = \frac{h}{3} [y_0 + 4y_1 + y_2]$$

$$= \frac{0.5}{3} [1 + 4 \times 0.866 + 0]$$

$$= 0.744$$

//

Q Apply Simpson's $1/3$ Rule to calculate $\int_0^{\pi} \sin x dx$ with $k=6$.

Sol: $h = \frac{\pi - 0}{6} = \pi/6, x_0 = 0, x_1 = \pi$

$$f(x_0) = f(x_0 + ph) = f(0 + 0 \times h) = 0$$

$$f(x_1) = f(0 + \pi/6) = f(\pi/6) = 1/2 = 0.5$$

$$f(x_2) = f(0 + 2\pi/6) = f(\pi/3) = \sqrt{3}/2 = 0.866$$

$$f(x_3) = f(0 + 3\pi/6) = f(\pi/2) = 1$$

$$f(x_4) = f(0 + 4\pi/6) = f(2\pi/3) = 0.866$$

$$f(x_5) = f(0 + 5\pi/6) = f(5\pi/6) = 0.5$$

$$f(x_6) = f(0 + 6\pi/6) = f(\pi) = 0$$

Now,

$$\int_0^{\pi} y dx = h/3 [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6]$$

$$= \frac{\pi}{6 \times 3} [0 + 4 \times 0.5 + 2 \times 0.866 + 4 \times 1 + 2 \times 0.866 + 4 \times 0.5 + 0]$$

$$= \frac{11.464}{1.8} \pi = 1.910 \pi$$

$$= 2.0021$$

Simpson's 3/8 Rule:

Setting $n=3$ in general formula, we observe all the differences higher than the third will become zero and we obtain.

$$\int_{x_0}^{x_3} y \cdot dx = 3h \left(y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right)$$

$$= 3h/8 (y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_3}^{x_6} y \cdot dx = 3h/8 (y_3 + 3y_4 + 3y_5 + y_6)$$

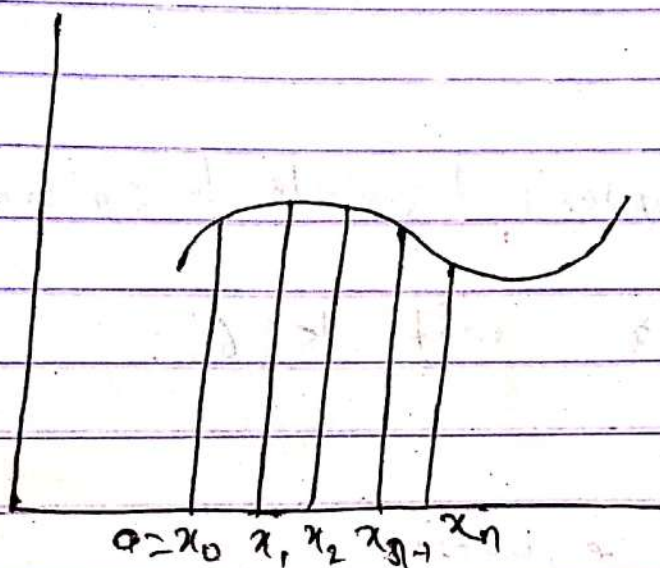
$$\int_{x_{n-3}}^{x_n} y \cdot dx = 3h/8 (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

On summation,

$$\int_{x_0}^{x_n} y \cdot dx = \frac{3h}{8} (y_0 + 3y_1 + 3y_2 + 3y_3 + 3y_4 + 3y_5 + 2y_6 + 3y_{n-2} + 3y_{n-1} + y_n)$$

Composite Trapezoidal Rule:

If the area to be integrated is large the trapezoidal rule can be improved by dividing the interval (a, b) into a number of small intervals and applying the rule. The sum of areas of all the sub-intervals is the integral of the interval (a, b) . This approach is known as composite multi segments approach.



$$h = \frac{b-a}{n}$$

$$x_i = x_0 + i^{\circ} h, \quad i = 0, 1, 2, 3, \dots, n$$

Area of the sub interval with the nodes

x_{i-1} and x_i is given by.

$$I_i = \int_{x_{i-1}}^{x_i} p_i(x) dx = h_{1/2} [f(x_{i-1}) + f(x_i)]$$

The total area of $n-1$ segment.

$$= h/2 [f(x_0) + f(x_1)] + h/2 [f(x_1) + f(x_2)] + \dots + h/2 [f(x_{n-1}) + f(x_n)]$$

$$= h/2 [f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n)]$$

which is the general form of composite trapezoidal.

Repeat

Applying the Simpson's $1/3$ Rule to calculate.

$$\int_0^{\pi} \sin x \, dx \quad \text{with } k=6.$$

Sol.

Here given equation

$$\int_0^{\pi} \sin x \, dx \quad \text{with } k=6$$

$$h = \frac{\pi - 0}{6} = \pi/6, \quad x_0 = 0, \quad x_1 = \pi$$

$$f(x_0) = f(x_0 + h) = f(0 + 0 \times h) = 0$$

$$f(x_1) = f(x_1 + h) = f(0 + \pi/6) = 0.5$$

$$f(x_2) = f(x_1+h) = f(0+\pi/6) = 0.866$$

$$f(x_3) = f(x_1+h) = f(0+2\pi/6) = 1$$

$$f(x_4) = f(x_1+h) = f(0+3\pi/6) = 0.866$$

$$f(x_5) = f(x_1+h) = f(0+4\pi/6) = 0.5$$

$$f(x_6) = f(x_1+h) = f(0+5\pi/6) = 0$$

Now,

we know that

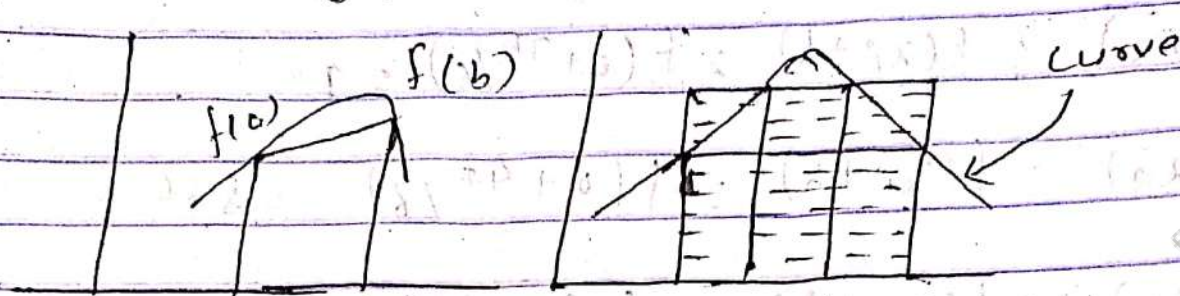
$$\begin{aligned} \int_0^{\pi} y \, dx &= h/3 [y_0 + y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6] \\ &= \frac{\pi}{3 \times 6} [0 + 4 \times 0.5 + 2 \times 0.866 + 4 \times 1 + 2 \times 0.866 + 4 \times 0.5 + 0] \end{aligned}$$

$$= \frac{11.464}{1.8}$$

$$= 2.0001$$

//

Gaussian Integration



$$I = \int_{-1}^1 f(x) dx$$

$$= c_1 f(x_1) + c_2 f(x_2)$$

we consider 3rd order polynomial as,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\int_{-1}^1 f(x) dx = \int_{-1}^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_{-1}^1$$

$$\int_{-1}^1 f(x) dx = 2a_0 + \frac{2}{3}a_2 \quad \text{--- (1)}$$

The formula given as,

$$\int_{-1}^1 f(x) dx = c_1 f(x_1) + c_2 f(x_2)$$

$$= C_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + C_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

$$= a_0(C_1 + C_2) + a_1(C_1 x_1 + C_2 x_2) + a_2(C_1 x_1^2 + C_2 x_2^2) + a_3(C_1 x_1^3 + C_2 x_2^3) \quad (2)$$

From eqⁿ (1) and (2)

$$C_1 + C_2 = 2$$

$$C_1 x_1 + C_2 x_2 = 0$$

$$C_1 x_1^2 + C_2 x_2^2 = 2/3$$

$$C_1 x_1^3 + C_2 x_2^3 = 0$$

Solving these four simultaneous eqⁿ is,

$$C_1 = C_2 = 1$$

$$x_1 = -1/\sqrt{3} \text{ and } x_2 = +1/\sqrt{3}$$

$$= -0.5773502, \quad = 0.5773502$$

Hence,

$$\int_{-1}^1 f(x) dx = C_1 f(x_1) + C_2 f(x_2) \\ = f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

Here, we use two points D_1 and D_2 . So, it is called two point quadrature formula. We can also deduced for higher points and for n -points, the generalized formula is,

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^n C_i f(x_i)$$

For $n=3$,

$$c_1 = 0.55556, \quad c_2 = 0.8889, \quad c_3 = 0.55556$$
$$x_1 = -0.7746, \quad x_2 = 0, \quad x_3 = 0.7746$$

Q Compute the integral $\int_{-2}^2 e^{-x/2} dx$ using Gaussian two point formula.

solⁿ

$$a = -2, \quad b = 2$$

$$z_1 = -0.57735, \quad z_2 = 0.57735$$

Thus,

$$x_1 = \left(\frac{b-a}{2}\right) * z_1 + \left(\frac{b+a}{2}\right)$$

$$= 2 * (-0.57735 + 0)$$

$$= -1.1547$$

$$x_2 = \left(\frac{b-a}{2}\right) * z_2 + \left(\frac{b+a}{2}\right)$$

$$= 2 * 0.57735 + 0$$

$$= 1.1547$$

now,

$$V = \frac{b-a}{2} \{ f(x_1) + f(x_2) \}$$

$$= 2 \{ e^{-1.1547/2} + e^{-1.1547/2} \}$$

$$= 4.6854 //$$

Q Compute the integral $\int_0^1 e^{-x^2} dx$ using Gaussian three point formula.

Solⁿ:

$$a=0, \quad b=1$$

$$z_1 = -0.7746, \quad z_2 = 0, \quad z_3 = 0.7746$$

$$x_1 = \left(\frac{b-a}{2}\right) * z_1 + \left(\frac{b+a}{2}\right)$$

$$= \left(\frac{1-0}{2}\right) * (-0.7746) + \left(\frac{0+1}{2}\right)$$

$$= 0.1127$$

$$x_2 = \left(\frac{b-a}{2}\right) * z_2 + \left(\frac{b+a}{2}\right)$$

$$= \left(\frac{1-0}{2}\right) * 0 + \left(\frac{1+0}{2}\right)$$

$$= 0.5$$

$$x_3 = \left(\frac{b-a}{2}\right) * z_3 + \left(\frac{b+a}{2}\right)$$

$$= \left(\frac{1-0}{2}\right) * (0.7746) + \left(\frac{1+0}{2}\right)$$

$$= 0.8873$$

Gaussian 3 point formula is given by,

$$\int_0^1 f(x) dx = \frac{b-a}{2} \left\{ c_1 * f(x_1) + c_2 * f(x_2) + c_3 * f(x_3) \right\}$$

$$f(x_1) = e^{(-0.1127)^2} = 0.98737$$

$$f(x_2) = e^{(-0.5)^2} = 0.7788$$

$$f(x_3) = e^{(-0.8873)^2} = 0.45507$$

$$C_1 * f(x_1) = 0.55556 * 0.98737 \\ = 0.54854$$

$$C_2 * f(x_2) = 0.88889 * 0.7788 \\ = 0.69227$$

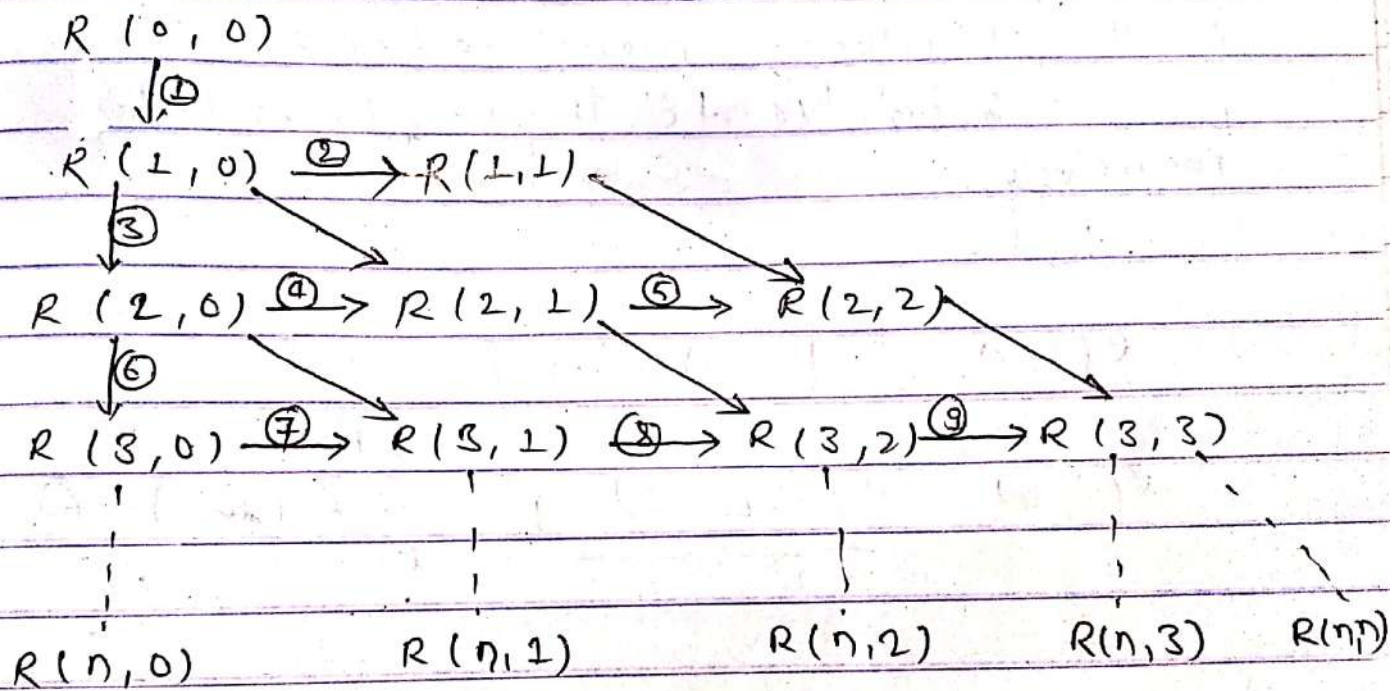
$$C_3 * f(x_3) = 0.55556 * 0.45507 \\ = 0.25281$$

Hence,

$$\int_0^1 e^{-x^2} dx = \frac{1-0}{2} \{ 0.54854 + 0.69227 + 0.25281 \} \\ = 0.74682 //$$

Romberg Integration:

This Romberg algorithm produces a triangular array of numbers all of which are numerical estimates of the definite integral $\int_a^b f(x) dx$. The array is denoted by the notation.



The elements of matrix are computed row by row in the order indicated in the figure. The circled numbers indicate the order of the computations and arrows indicates the dependences of element. An element at the head $\rightarrow$ depends on the elements at the tail end.

The first column of this matrix contains estimates of the integral obtain by recursive trapezoid

formula with decreasing value of step size. Explicitly, $R(n, 0)$ is the result of applying the trapezoidal with 2^n equal sub-intervals.

The first term of them, $R(0, 0)$ is obtained just with one trapezoid. Similarly $R(1, 0)$ is obtained with trapezoids and so on. Elements in the 1st column represent the trapezoidal rule at $h, h/2, h/4$ etc. They can be evaluated recursively.

$$h = b - a$$

$$R(0, 0) = h/2 [f(a) + f(b)]$$

$$R(i, 0) = \frac{R(i-1, 0)}{2} + h_i \sum_{k=1}^{2^{i-1}} f(x_{k-1}) \quad \text{--- (1)}$$

$$i = 1, 2, 3, \dots, n$$

where,

$$h_i = \frac{b-a}{2^i}$$

$$x_k = a + kh_i$$

$$R_{i,j} = \frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1} \quad \text{--- (2)}$$

eqn (2) is called Romberg integration formula.

Compute the Romberg integration estimate $R_{2,2}$ for $\int_1^2 \frac{1}{x} dx$.

Sol

$$R(0,0) = h_{1/2} [f(a) + f(b)]$$

$$= \frac{1}{2} [1 + \frac{1}{2}]$$

$$= 0.75$$

$$R(1,0) = \frac{R(0,0) + h_1 \sum_{k=1}^L f(x_k)}{2}$$

$$= \frac{0.75}{2} + \frac{1}{2} \times \frac{1}{1.5}$$

$$x_k = a + kh_1$$

$$x_1 = 1 + 1 \times (0.5)$$

$$= 1.5$$

$$R(2,0) = \frac{R(1,0)}{2} + \frac{b-a}{2^2} \times \{f(x_2) + f(x_3)\}$$

$$x_k = a + kh_1$$

$$x_3 = a + 3h_2$$

$$= 1 + 3 \times \frac{1}{4}$$

$$= 1.75$$

$$R(2,0) = \frac{0.708333}{2} + \frac{1}{4} \times \left\{ \frac{1}{1.5} + \frac{1}{1.75} \right\}$$

$$= 0.6970237$$

$$R(1,1) = \frac{4^1 \times R(1,0) - R(0,0)}{4^1 - 1}$$

$$= \frac{4 \times 0.708333 - 0.75}{3}$$

$$= 0.6944444$$

$$R(2,1) = \frac{4^2 \times R(2,0) - R(1,0)}{4^2 - 1}$$

$$= \frac{4 \times 0.6970237 - 0.708333}{4^1 - 1}$$

$$= 0.6932538$$

$$R(2,2) = \frac{4^2 \times R(2,1) - R(1,1)}{4^2 - 1}$$

$$= \frac{16 \times 0.6932538 - 0.6944444}{15}$$

$$= 0.6931744 //$$

$$\text{True value} = [\log x]_1^2 = \int_1^2 \frac{1}{x} dx$$

$$= \log 2 - \log 1$$

$$= 0.6931471 - 0$$

$$= 0.6931471$$

$$\text{Error} = \text{True Value} - \text{Approximate value}$$

$$= 0.6931471 - 0.6931744$$

$$= 0.00002733 //$$

Solution of Ordinary differential equation.

Differential equation:

Mathematical models, which uses differential calculus to express relationship between variable are known as differential equation. The differential equation may appear in different forms.

Number of independent variable?

The quantity being differentiated is called the dependent variables and the quantity with respect to which the dependent variable is differentiated is called independent variables.

If there is one independent variable the equation is called an ordinary equation. If there is more than one independent variable the equation is called partial differential equation.

E.g.:

$$\left. \begin{aligned} \frac{dv}{dt} &= -\frac{k}{m} v^2 \\ \frac{dm}{dt} - km &= 0 \end{aligned} \right\} \text{ordinary}$$

$$\left. \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial u^2} = f(x, y) \right\} \text{partial differential equation.}$$

Order of equation

The order of the differential equation is the highest derivative that appears in the equation. When the equation only contains a first derivative, it is called a first order differential equation. If the highest derivative is second derivative, the equation is called a second order differential equation.

$$\frac{dy}{dx} = f(x, y) \rightarrow \text{order-1}$$

$$y'' = f(x, y, y') \rightarrow \text{order-2}$$

$$m \frac{d^2 y}{dt^2} + \frac{x dy}{dt} + ky = 0 \rightarrow \text{order-2}$$

Degree of equation:

The degree of a differential equation is the power of the highest order derivative.

eg:

$$(i) \quad xy' + y^2y' = 2y + 3 \rightarrow \text{degree-1, order-2}$$

$$(ii) \quad (y''')^2 + 5y' = 0 \rightarrow \text{degree-2, order-3}$$

Initial Value problem vs boundary value problem:

The solution of an ordinary differential equation requires auxiliary condition. For n -order equation, n conditions are required. If all the conditions are specified at the same value of the independent variable then the problem is called initial value problem. Eg. The eqⁿ $y' = x^2 + y^2$, given $y(0) = 1$.

If the conditions are known at different values of the independent variable usually at the extreme point or boundary of system, then problem is known as boundary value problems.

eg:

The eqⁿ $y' = y$, given $y(0) + y(2) = 2$.

Solving Initial Value problems:

① Taylor series:

We expand a function $y(x)$ at a point $x = x_0$ using Taylor series expansion.

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \dots + \frac{(x - x_0)^n}{n!}y^{(n)}(x_0) + \dots$$

Q Solve $y' = x^2 + y^2$ under the condition $y(x) = 1$ when $x_0 = 0$.

Solⁿ

$$y' = x^2 + y^2$$

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2 \left[y \frac{dy'}{dx} + y' \frac{dy}{dx} \right]$$

$$= 2 + 2 (yy'' + y'^2)$$

$$= 2 + 2yy'' + 2y'^2$$

at $x_0 = 0$ and $y(0) = 1$

$$y'(0) = 0^2 + 1^2 = 1$$

$$y''(0) = 2 \times 0 + 2 \times 1 \times 1 = 2$$

$$y'''(0) = 2 + 2 \times 1 \times 2 + 2 \times 1^2 = 8$$

Substituting above values in the Taylor series,

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \dots$$

$$= 1 + 2 \cdot 1 + \frac{x^2 \cdot 2}{2!} + \frac{x^3 \cdot 8}{3!}$$

$$= 1 + 2 + x^2 + \frac{4x^3}{3} //$$

Q Solve the differential equation $y' = 3x^2$ such that $y = 1$ at $x = 1$. Find y for $x = 2$ by using 1st 4 - terms.

solⁿ,

$$y = 1, x_0 = 1$$

$$y' = 3x^2$$

$$y'' = 6x$$

$$y''' = 6$$

at $x_0 = 1$ and $y(0) = 1$

$$y'(1) = 3 \times 1^2 = 3$$

$$y''(1) = 6 \times 1 = 6$$

$$y'''(1) = 6$$

Substituting the value in Taylor series,

$$y(x) \approx y(x_0) + (x-x_0)y' + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

$$= 1 + (2-1)3 + \frac{(2-1)^2 \times 6}{2!} + \frac{(2-1)^3 \times 6}{3!}$$

$$= 1 + 3 + \frac{6}{2} + \frac{6}{6}$$

$$= 1 + 3 + 3 + 1$$

$$= 8 \quad //$$

9 $y' = x^2 + y^2$ for $x \geq 0.25$, $y(0) = 1$.

Solⁿ

$$y' = x^2 + y^2$$

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2 \left[y \frac{dy'}{dx} + y' \frac{dy}{dx} \right]$$

$$= 2 + 2 (y \cdot y'' + y' \cdot y')$$

$$= 2 + 2y \cdot y'' + 2(y')^2$$

at $x_0 = 0$ and $y(0) = 1$

$$y'(0) = 0^2 + 1^2 = 1$$

$$y''(0) = 2 \times 0 + 2 \times 1 \times 1 = 2$$

$$y'''(0) = 2 + 2 + 1 + 2 + 2 \times 1^2 = 8$$

substituting,

$$y = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2 y''(x_0)}{2!} + \frac{(x - x_0)^3 y'''(x_0)}{3!}$$

$$= 1 + (0.25 - 0) \times 1 + \frac{(0.25 - 0)^2 \times 2}{2!} + \frac{(0.25 - 0)^3 \times 8}{3!}$$

$$= 1.333$$

//

② Picard's Method:

Consider the differential equation:

$$\frac{dy}{dx} = f(x, y)$$

We can integrate this equation to obtain the solution in the interval (x_0, x)

$$\int_{x_0}^x dy = \int_{x_0}^x f(x, y) dx$$

$$\text{or, } y(x) - y(x_0) = \int_{x_0}^x f(x, y) dx$$

$$\text{or, } y(x) = y(x_0) + \int_{x_0}^x f(x, y) dx$$

Since, y appears under the integral sign on the right side, the integration cannot be performed. The dependent variable y should be replaced by either a constant or $f(x)$.

Since, we know that initial value of (at $x = x_0$) we may use this as a first approximation to the solution and result can be used on right hand side to obtain the next approximation.

$$y' = f_0 + \int_{x_0}^x f(x, y_0) dx$$

In general,

$$y^n = y + \int_{x_0}^x f(x, y_{n-1}) dx. \text{ This equation is called Picard's method.}$$

Q. Solve the eqⁿ $y' = x^2 + y^2$ by using Picard's method with the initial condition $y(0) = 0$. Find the value of $y(0.2)$ correct upto 2nd approximation.

Solⁿ!

$$y' = x^2 + y^2$$

$$y_0 = 0, \quad x_0 = 0$$

$$y^1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$y^1 = 0 + \int_0^x (x^2 + (y_0)^2) dx$$

$$= \int_0^x (x^2 + 0) dx$$

$$= x^3/3$$

$$y^2 = y_0 + \int_{x_0}^x f(x, y^1) dx$$

$$= 0 + \int_0^x (x^2 + (x^3/3)^2) dx$$

$$= \int_0^x (x^2 + x^6/9) dx$$

$$= 1/9 x^0$$

$$= \frac{1}{9} \left[3x^3 + \frac{27}{7} \right]$$

$$= \frac{1}{9} (3x^3 + \frac{27}{7})$$

$$y(x) = \frac{1}{9} (3x^3 + \frac{27}{7})$$

$$y(0.2) = \frac{1}{9} (3 \times (0.2)^3 + \frac{(0.2)^7}{7})$$

$$= 0.0006667 //$$

Q. $\frac{dy}{dx} = y + x$, $y(0) = 1$ into 5 times.

Solⁿ: $y^1 = y_0 + \int_{x_0}^x f(x, y_0) dx$
 $= 1 + \int_0^x (y_0 + x) dx$

$$= 1 + \int_0^x (1 + x) dx$$

$$= 1 + x + \frac{x^2}{2}$$

$$y^2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

$$= 1 + \int_0^x (y_1 + x) dx$$

$$= 1 + \int_0^x (1 + x + \frac{x^2}{2} + x) dx$$

$$= 1 + \left(2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} \right)$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

$$y^3 = y_0 + \int_{x_0}^x f(x, y_2) dx$$

$$= 1 + \int_0^x (y_2 + x) dx$$

$$= 1 + \int_{x_0}^{x_6} (1 + x + x^2 + x^3/6 + x) dx$$

$$= 1 + x + x^2 + x^3/3 + x^4/24$$

$$y^4 = y_0 + \int_{x_0}^x f(x, y_3) dx$$

$$= 1 + \int_{x_0}^x (y_3 + x) dx$$

$$= 1 + \int_{x_0}^x (1 + x + x^2 + x^3/3 + x^4/24 + x) dx$$

$$= 1 + x + x^2 + x^3/3 + x^4/12 + x^5/120$$

$$y^5 = y_0 + \int_{x_0}^x f(x, y_4) dx$$

$$= y_0 + \int_{x_0}^x (y_4 + x) dx$$

$$= 1 + \int_{x_0}^x (1 + x + x^2 + x^3/3 + x^4/12 + x^5/120 + x) dx$$

$$= 1 + x + x^2 + x^3/3 + x^4/12 + x^5/60 + x^6/720 //$$

③ Euler's Method:

Consider the ordinary differential equation,
 $\frac{dy}{dx} = f(x, y) \quad , \quad y(0) = y_0 \quad \text{--- (1)}$

We wish to solve eqⁿ (1), for the value of y at $x = x_1$

where,

$$x_i = x_0 + i^{\text{th}} h \quad , \quad i = 0, 1, 2, \dots$$

Now,

integrate the eqⁿ (1), we get

$$\int_{x_0}^{x_1} y' = \int_{x_0}^{x_1} f(x, y) dx$$

$$\Rightarrow y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y) dx \quad \text{--- (2)}$$

Assuming that $f(x, y) \approx f(x_0, y_0)$ in the range $x_0 \leq x \leq x_1$

$$y(x_1) = y(x_0) + f(x_0, y_0) \int_{x_0}^{x_1} dx$$

$$= y(x_0) + f(x_0, y_0) (x_1 - x_0)$$

$$= y(x_0) + h f(x_0, y_0)$$

Similarly, for the range $x_1 \leq x \leq x_2$ eqⁿ ② can be written as,

$$y(x_2) \approx y(x_1) + hf(x_1, y_1)$$

Generating this we get,

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i) //$$

Q Approximate the solution of the initial value problem $y' = 2x + y$, $y(0) = 1$ by using Euler's method with step size of 0.1. Approximate the value of $y(0.4)$.

Solⁿ

$$f(x, y) = 2x + y$$

$$y(0) = 1$$

$$h = 0.1$$

$$y(0.4) = ?$$

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i)$$

Iteration 1st

$$x_0 = 0, y_0 = 1$$

$$\begin{aligned}
 y(x_1) &= y(0.1) \\
 &= y(0) + 0.1 * 1 \\
 &= 1 + 0.1 \\
 &= 1.1
 \end{aligned}$$

Iteration 2,

$$x_1 = 0.1, y_1 = 1.1,$$

$$\begin{aligned}
 y(x_2) &= y(0.2) \\
 &= y(0.1) + 0.1 * f(2 * 0.1 + 1.1) \\
 &= 1.1 + 0.1 * f(0.2 + 1.1) \\
 &= 1.23
 \end{aligned}$$

Iteration 3,

$$x_1 = 0.2, y_1 = 1.23$$

$$\begin{aligned}
 y(x_3) &= y(0.3) \\
 &= y(0.2) + 0.1 * f(2 * 0.2 + 1.23) \\
 &= 1.393
 \end{aligned}$$

Iteration 4,

$$x_1 = 0.3, y_1 = 1.393$$

$$y(x_4) = y(0.4)$$

$$= y(0.3) + 0.1 * f(2 * 0.3 + 1.393)$$

$$= 1.5923 //$$

Q

Approximate the solution of the initial value problem $y' = x^2 + y^2$, $y(0) = 1$ by using Euler's method with step size of 0.2. Approximate the value of $y(0.6)$.

Sol:

$$y' = x^2 + y^2$$

$$y(0) = 1$$

$$x_0 = 0$$

$$h = 0.2$$

formula,

$$y(x_{i+1}) = y(x_i) + h f(x_i, y_i)$$

Iteration 1,

$$x_0 = 0, y(0) = 1$$

$$y(x_1) = y(0.2)$$

$$= y(0) + h f(x_0, y_0)$$

$$= 1 + 0.2 * (0^2 + 1^2)$$

$$= 1.2$$

Iteration 2,

$$x_0 = 0.2, y_0 = 1.2$$

$$\begin{aligned} y(x_2) &= y(0.4) \\ &= y(0.2) + 0.2 * f(0.2^2 + 1.2^2) \\ &= 1.2 + 0.2 * (0.2^2 + 1.2^2) \\ &= 1.496 \end{aligned}$$

Iteration 3,

$$x_0 = 0.4, y_0 = 1.496$$

$$\begin{aligned} y(x_3) &= y(0.6) \\ &= y(0.4) + 0.2 * f(x_0, y_0) \\ &= 1.496 + 0.2 * (0.4^2 + 1.496^2) \\ &= 1.976 // \end{aligned}$$

④ Run's Method;

$$y' = 2x + y, \quad y(0) = 1, \quad h = 0.1, \quad y(0.4) = ?$$

sol:

Iteration 1st,

$$i = 0, \quad x_0 = 0, \quad y_0 = 1$$

$$w_1 = f(x_0, y_0) = 2 \times 0 + 1 = 1$$

$$w_2 = f(x_0 + h, y_0 + h \times w_1)$$

$$= f(0 + 0.1, 1 + 0.1 \times 1)$$

$$= f(0.1, 1.1)$$

$$= 1.3$$

$$y(x_1) = y(x_0) + h/2 (w_1 + w_2)$$

$$= 1 + 0.1/2 (1 + 1.3)$$

$$y(0.1) = 1.115$$

Iteration 2nd,

$$i = 1, \quad x_1 = 0.1, \quad y_1 = 1.115$$

$$w_1 = f(x_1, y_1) = f(0.1, 1.115)$$

$$= 1.315$$

$$m_2 = f(x_1 + h, y_1 + h * m_1)$$

$$= f(0.1 + 0.1, 1.115 + 0.1 * 1.315)$$

$$= f(0.2, 1.2465)$$

$$= 1.6465$$

$$y(x_2) = y(x_1) + h/2 (m_1 + m_2)$$

$$= 1.115 + 0.1/2 (1.315 + 1.6165)$$

$$y(0.2) = 1.263$$

Iteration 3rd,

$$p = 2, \quad x_2 = 0.2, \quad y_2 = 1.263$$

$$m_1 = f(x_2, y_2) = f(0.2, 1.263)$$

$$= 1.663$$

$$m_2 = f(x_2 + h, y_2 + h * m_1)$$

$$= f(0.2 + 0.1, 1.263 + 0.1 * 1.663)$$

$$= 2.0299$$

$$y(x_3) = y(x_2) + h_2 (m_1 + m_2)$$

$$= 1.263 + 0.1/2 (2.663 + 2.8293)$$

$$y(0.3) = 1.447615$$

Iteration 4th,

$$i = 3, x_3 = 0.3, y_3 = 1.447615$$

$$m_1 = f(x_3, y_3) = f(0.3, 1.447615)$$

$$= 2.047615$$

$$m_2 = f(x_3 + h, y_3 + h * m_1)$$

$$= f(0.3 + 0.1, 1.447615 + 0.1 * 2.047615)$$

$$= 2.4523765$$

$$y(x_4) = y(x_3) + h_2 (m_1 + m_2)$$

$$= 1.447615 + 0.1/2 (2.047615 + 2.4523765)$$

$$= 1.672614575 //$$

Q Approximate the solution of the initial value problem $y' = x^2 + y$, $y(0) = 1$, by using Heun's method with step size of 0.05. Approximate the value of $y(0.2)$.

Solⁿ:

here,

$$h = 0.05$$

Iteration 1st,

$$i = 0, \quad x_0 = 0, \quad y_0 = 1$$

$$m_1 = f(x_0, y_0) = f(0, 1) = 1$$

$$m_2 = f(x_0 + h, y_0 + h * m_1)$$

$$= f(0 + 0.05, 1 + 0.05 * 1)$$

$$= 1.0525$$

$$y(x_1) = y(x_0) + h_2 (m_1 + m_2)$$

$$= 1 + \frac{0.05}{2} (1 + 1.0525)$$

$$y(0.05) = 1.0513125$$

Iteration 2nd,

$$i = 1, \quad x_1 = 0.05, \quad y_1 = 1.0513125$$

$$m_1 = f(x_1, y_1) = f(0.05, 1.0513125) \\ = 1.0538125$$

$$m_2 = f(x_1 + h, y_1 + h * m_1)$$

$$= f(0.05 + 0.05, 1.0513125 + 0.05 * 1.0538125)$$

$$= 1.1124$$

$$y(x_2) = y(x_1) + h y_2 (m_1 + m_2)$$

$$= 1.0513125 + \frac{0.05}{2} (1.0538125 + 1.1124)$$

$$y(0.1) = 1.1055$$

Iteration 3rd,

$$i = 2, \quad x_2 = 0.1, \quad y_2 = 1.1055$$

$$m_1 = f(x_2, y_2) = f(0.1, 1.1055)$$

$$= 1.1155$$

$$m_2 = f(x_2 + h, y_2 + h * m_1)$$

$$= f(0.2 + 0.05, 1.1055 + 0.05 * 1.1155)$$

$$= 1.18377$$

$$y(x_3) = y(x_2) + h/2 (m_1 + m_2)$$

$$= 1.1055 + 0.05/2 (1.1155 + 1.18377)$$

$$y(0.15) = 1.16298$$

Iteration 4th,

$$i = 3, \quad x_3 = 0.15, \quad y_3 = 1.16298$$

$$m_1 = f(x_3, y_3)$$

$$= f(0.15, 1.16298)$$

$$= 1.18548$$

$$m_2 = f(x_3 + h, y_3 + h * m_1)$$

$$= f(0.15 + 0.05, 1.16298 + 0.05 * 1.18548)$$

$$= 1.262254$$

$$y(x_4) = y(x_3) + h/2 (m_1 + m_2)$$

$$= 1.16298 + 0.05/2 (1.18548 + 1.262254)$$

$$y(0.2) = 1.22417835 //$$

⑤ Runge . Kutta Method :

formula :

$$m_1 = f(x, y)$$

$$m_2 = f(x + h/2, y + h/2 * m_1)$$

$$m_3 = f(x + h/2, y + h/2 * m_2)$$

$$m_4 = f(x + h, y + h * m_3)$$

$$y(x_{i+1}) = y(x_i) + h/6 (m_1 + 2m_2 + 2m_3 + m_4)$$

Q $f(x, y) = 2x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.4$

Solⁿ :

$$m_1 = f(0, 1)$$

$$= 1$$

$$m_2 = f(0 + 0.4/2, 1 + 0.4/2 * 1)$$

$$= 1.6$$

$$m_3 = f(0 + 0.4/2, 1 + 0.4/2 * 1.6)$$

$$= 1.72$$

$$m_4 = f(0 + 0.4, 1 + 0.4 * 1.72)$$

$$= 2.488$$

$$y(x_1) = y(0.4)$$

$$= y(x_0) + h/6 (m_1 + 2m_2 + 2m_3 + m_4)$$

$$= 1 + 0.4/6 (1 + 2 * 1.6 + 2 * 1.72 + 2.488)$$

$$= 1.6752 //$$

Q Apply the Runge Kutta Method of 2nd order and 4th order to find an approximate value of y when $h = 0.2$ given that $dy/dx = x+y$ in $y(0) = 1$

Solⁿ

$$x_0 = 0, y_0 = 1,$$

$$h = 0.2$$

$$y' = x+y$$

For 2nd order,

$$m_1 = f(x, y) = f(0, 1) = 1$$

$$m_2 = f(0 + 0.2, 1 + 0.2 \times 1) = 1.4$$

$$y(x_1) = y(x_0) + 0.2/2 (1 + 1.4)$$

$$= 1 + 0.1 (1 + 1.4)$$

$$= 1.24$$

For 4th order,

$$m_1 = f(0, 1) = 1$$

For 2nd order
$m_1 = f(x, y)$
$m_2 = f(x+h, y+h \times m_1)$
$y(x_{i+1}) = y(x_i) + h_2(m_1 + m_2)$

$$m_2 = f(0 + 0.2/2, 1 + 0.2/2 * 1)$$

$$\approx 1.2$$

$$m_3 = f(0 + 0.2/2, 1 + 0.2/2 * 1.2)$$

$$\approx 1.22$$

$$m_4 = f(0 + 0.2, 1 + 0.2 * 1.22)$$

$$\approx 1.444$$

$$y(x_1) = y(x_0) + 0.2/6 (m_1 + 2m_2 + 2m_3 + m_4)$$

$$= 1 + \frac{0.2}{6} (1 + 2 * 1.2 + 2 * 1.22 + 1.444)$$

$$= 1.2428 //$$

Q Use the 4th order Runge Kutta method with step size 0.2 to estimate $y(0.4)$ of

$$y' = x^2 + y^2, \quad y(0) = 0.$$

Sol

$$m_1 = f(0, 0) \\ = 0$$

$$m_2 = f\left(0 + \frac{0.2}{2}, 0 + \frac{0.2}{2} * 0\right) \\ = 0.01$$

$$m_3 = f\left(0 + \frac{0.2}{2}, 0 + \frac{0.2}{2} * 0.01\right) \\ = 0.01 + 0.000001 \\ = 0.010001$$

$$m_4 = f\left(0 + 0.2, 0 + 0.2 * 0.010001\right) \\ = 0.0100040008$$

$$y(x_1) = y(0.4)$$

$$= y(x_0) + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$= 0 + \frac{0.2}{6}(0 + 2 * 0.01 + 2 * 0.010001 + 0.0100040008)$$

$$= 0.009666866693$$

Solving system of linear equation.

Basic Gauss Elimination Method:

The approach is design to solve a general set of n equation and n unknowns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

Gaussian elimination of unknowns:

1) Forward elimination of unknowns.

→ In this step, the unknowns eliminated in each starting with the first equation.

→ This way the equation are reduced to one equation and one unknown in each equation.

2) Back substitution.

→ In this step, starting from the least equation each of the unknown is found.

$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &= 13 \\ x_1 + 3x_2 + 2x_3 &= 9 \\ 3x_1 + 6x_2 + x_3 &= 16 \end{aligned}$$

Solⁿ

Converting these system of linear equation in matrix form.

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 13 \\ 1 & 3 & 2 & 9 \\ 3 & 6 & 1 & 16 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 13 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & -7 & -7 \end{array} \right]$$

Now,

the equivalent solution are

$$2x_1 + 4x_2 + 3x_3 = 13 \quad \text{--- (I)}$$

$$2x_2 + x_3 = 5 \quad \text{--- (II)}$$

$$-7x_3 = -7 \quad \text{--- (III)}$$

Here,

from eqⁿ (III),
 $x_3 = 1$

from (ii),

$$2x_2 + 1 = 5$$

$$x_2 = 2$$

from (i),

$$2x_1 + 4 \times 2 + 3 \times 1 = 13$$

$$x_1 = 1$$

Hence,

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 1$$

||

Q. $2x_1 + x_2 + x_3 = 5$
 $4x_1 + 6x_2 = -2$
 $-2x_1 + 7x_2 + 2x_3 = 9$

solⁿ,

Converting these system of linear equation in matrix form,

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & 6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 4 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array} \right] R_3 \rightarrow R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 4 & -2 & -12 \\ 0 & 0 & 7 & 38 \end{array} \right]$$

Now,

the equivalent solution are,

$$2x_1 + x_2 + x_3 = 5 \quad \leftarrow \textcircled{I}$$

$$4x_2 - 2x_3 = -12 \quad \leftarrow \textcircled{II}$$

$$7x_3 = 38 \quad \leftarrow \textcircled{III}$$

from eqⁿ (iii),

$$x_3 = 38/7$$

from eqⁿ (ii),

$$4x_2 - 2 \times 38/7 = -12$$

$$4x_2 = -12 + 76/7$$

$$4x_2 = -8/7$$

$$x_2 = -2/7$$

from eqⁿ (i),

$$4x_1 + (-2/7) + (38/7) = 5$$

$$2x_1 = 5 + 2/7 - 38/7$$

$$2x_1 = -1/7$$

$$x_1 = -1/14$$

Hence,

$$x_1 = -1/14$$

$$x_2 = -2/7$$

$$x_3 = 38/7$$

//

Gauss Jordan Method :

- i) It is variation of Gauss elimination method.
- ii) The elimination step results the identity matrix rather than a triangular matrix. Therefore, the backsubstitution is not necessary.
- iii) All techniques developed for Gauss elimination are still valid for Gauss Jordan elimination.
- iv) The goal in Gauss Jordan elimination is to use row operations (Inter-change two rows, multiply a row by non-zero constant, add two rows or multiply one row to another row) to change the augmented matrix to row reduced echelon form which looks like following matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array} \right]$$

- v) To get a matrix of this form i.e. row reduced to echelon form, the steps are given below :

- Use row operations to make all 1.
Then make other element in column 1st is 0.

- Use Row operation to make $a_{22} = 1$. Then make other element in column 2nd is 0.
- Use Row operation to make $a_{33} = 1$. Then make other element in column 3rd is 0.
- Now we obtain the value of unknown directly from the Right hand side of augmented matrix.

9

$$2x_1 - x_2 + 4x_3 = 15$$

$$2x_1 + 3x_2 - 2x_3 = 1$$

$$3x_1 + 2x_2 - 4x_3 = -4$$

Solⁿ,

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 15 \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array} \right] \quad R_1 \rightarrow \frac{1}{2} R_1$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & \frac{15}{2} \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & \frac{15}{2} \\ 0 & 4 & -6 & -14 \\ 0 & \frac{7}{2} & -10 & -\frac{53}{2} \end{array} \right] \quad R_2 \rightarrow \frac{1}{4} R_2$$

$$\left[\begin{array}{ccc|c} 1 & -4/2 & 2 & 15/2 \\ 0 & 1 & -3/2 & -7/2 \\ 0 & 7/2 & -10 & -53/2 \end{array} \right] \begin{array}{l} R_1 \rightarrow R_1 + 1/2 R_2 \\ R_3 \rightarrow R_3 - 7/2 R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5/4 & 23/4 \\ 0 & 1 & -3/2 & -7/2 \\ 0 & 0 & -19/4 & -57/4 \end{array} \right] R_3 \rightarrow -4/19 R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5/4 & 23/4 \\ 0 & 1 & -3/2 & -7/2 \\ 0 & 0 & 1 & 3 \end{array} \right] \begin{array}{l} R_1 \rightarrow R_1 - 5/4 R_3 \\ R_2 \rightarrow R_2 + 3/2 R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Hence,

$$x_1 = 2$$

$$x_2 = 1$$

$$x_3 = 3$$

//

$$\begin{aligned}
 2x_1 + 4x_2 + 3x_3 &= 13 \\
 x_1 + 3x_2 + 2x_3 &= 9 \\
 3x_1 + 6x_2 + x_3 &= 16
 \end{aligned}$$

solⁿ,

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 13 \\ 1 & 3 & 2 & 9 \\ 3 & 6 & 1 & 16 \end{array} \right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 2 & 4 & 3 & 13 \\ 3 & 6 & 1 & 16 \end{array} \right] \quad \begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 3R_1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -2 & -1 & -5 \\ 0 & -3 & -5 & -11 \end{array} \right] \quad R_2 \rightarrow -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & -3 & -5 & -11 \end{array} \right] \quad \begin{aligned} R_1 &\rightarrow R_1 - 3R_2 \\ R_3 &\rightarrow R_3 + 3R_2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & \frac{13}{2} \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{7}{2} & -\frac{7}{2} \end{array} \right] \quad R_3 \rightarrow -\frac{2}{7}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & 1/2 & 5/2 \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{array}{l} R_1 \rightarrow R_1 - 1/2 R_3 \\ R_2 \rightarrow R_2 - 1/2 R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Hence,

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 1$$

//

Matrix inversion:

- Although Gauss Jordan method is more time consuming than Gauss elimination method. It provides simple approach for computing inverse of a matrix.
- A matrix X is said to be inverse of A if $AX = I$, where I is identity matrix. i.e. same order as that of matrix A .
- Matrix inversion can be computed by using Gauss Jordan method of following steps.

Step 1:

Augment the coefficient matrix with identity matrix as below

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right]$$

Step 2:

Apply Gauss Jordan method to the augmented matrix to reduce coefficient matrix into Identity matrix as below.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a'_{11} & a'_{12} & a'_{13} \\ 0 & 1 & 0 & a'_{21} & a'_{22} & a'_{23} \\ 0 & 0 & 1 & a'_{31} & a'_{32} & a'_{33} \end{array} \right]$$

Now R.H.S. of the augmented matrix is the inverse of the coefficient matrix.

Q Find the inverse of matrix $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$.

Solⁿ:

The augmented matrix is,

$$\left[\begin{array}{cc|cc} 7 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array} \right] R_1 \rightarrow \frac{1}{7} R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3/7 & 1/7 & 0 \\ 5 & 2 & 0 & 1 \end{array} \right] R_2 \rightarrow R_2 - 5R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3/7 & 1/7 & 0 \\ 0 & -1/7 & -5/7 & 1 \end{array} \right] R_2 \rightarrow -7R_2$$

$$\left[\begin{array}{cc|cc} 1 & 3/7 & 1/7 & 0 \\ 0 & 1 & 5 & -7 \end{array} \right] R_1 \rightarrow R_1 - 3/7 R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & -7 \end{array} \right]$$

Therefore the inverse of matrix A is

$$\begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix} //$$

Q.
$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

solⁿ:

The augmented matrix is,

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \quad R_2 \rightarrow R_2 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \quad R_2 \rightarrow \frac{1}{5}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 \rightarrow R_1 + R_2 \\ R_3 \rightarrow R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/5 & 3/5 & 1/5 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & -4/5 & 2/5 & 1 \end{array} \right] \quad R_3 \rightarrow 5R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/5 & 3/5 & 1/5 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array} \right] \quad \begin{array}{l} R_1 \rightarrow R_1 - 3/5R_3 \\ R_2 \rightarrow R_2 + 2/5R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array} \right]$$

Therefore, the inverse of matrix A is,

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

//

Matrix factorization:

→ LU decomposition (also called LU factorization) factors a matrix as the product lower triangular matrix and Upper triangular matrix.

where

$$L = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

L is also known as lower triangular matrix and U is called as upper triangular matrix.

Now, System of linear equations.

$AX = C$ can be written as

$$LUX = C \quad \therefore A = LU$$

Multiplying both sides with L^{-1} , we get

$$L^{-1} LUX = L^{-1} C$$

$$IUX = L^{-1} C$$

$$UX = L^{-1}C$$

Let

$$L^{-1}C = Z$$

then

$$LZ = C \quad \text{--- (i)}$$

$$UX = Z \quad \text{--- (ii)}$$

Now, we can solve system of equation in two steps.

(i) Solve eqⁿ (i) for Z by using forward substitution.

(ii) eqⁿ (ii) to calculate the solution vector X by back substitution.

* Do it's LU decomposition!

Coefficient matrix A of a system of linear equation can be decomposed into two triangular matrix L and U , such that,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & \dots & l_{2n} & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & \dots & v_{1n} \\ 0 & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_{nn} \end{bmatrix}$$

→ If L has 1's on its diagonal, then it is called Doolittle factorization.

→ Thus Doolittle algorithm assumes that $L_{11}=1, L_{22}=1, L_{nn}=1$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix}$$

$\underbrace{\hspace{10em}}_L \qquad \underbrace{\hspace{10em}}_U$

$$= \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ l_{21}v_{11} & l_{21}v_{12} + v_{22} & l_{21}v_{13} + v_{23} \\ l_{31}v_{11} & l_{31}v_{12} + l_{32}v_{22} & l_{31}v_{13} + l_{32}v_{23} + v_{33} \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Solⁿ

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Multiplying LU and Setting the answer to A,
we get

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

1st Row

$$u_{11} = 1$$

$$u_{12} = 2$$

$$u_{13} = 4$$

2nd row,

$$L_{21}V_{11} = 3$$

$$L_{21} \times 1 = 3$$

$$L_{21} = 3$$

$$L_{21}V_{12} + V_{22} = 8$$

$$3 \times 2 + V_{22} = 8$$

$$V_{22} = 2$$

$$L_{21}V_{13} + V_{23} = 14$$

$$3 \times 4 + V_{23} = 14$$

$$V_{23} = 2$$

3rd row,

$$L_{31}V_{11} = 2$$

$$L_{31} \times 1 = 2$$

$$L_{31} = 2$$

$$L_{31}V_{12} + L_{32}V_{22} = 6$$

$$2 \times 2 + L_{32} \times 2 = 6$$

$$2L_{32} = 2$$

$$L_{32} = 1$$

$$l_{31}V_{13} + l_{32}V_{23} + V_{33} = 13$$

$$2 \times 4 + 1 \times 2 + V_{33} = 13$$

$$V_{33} = 3$$

$$\begin{aligned} V_{11} &= 1, & l_{21} &= 3, & V_{22} &= 2 \\ V_{12} &= 2, & l_{31} &= 2, & V_{23} &= 2 \\ V_{13} &= 4, & l_{32} &= 1, & V_{33} &= 3 \end{aligned}$$

then

$$A = LU$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Q Solve the following system of eqⁿ by using Doolittle LU decomposition method.

Q

$$3x_1 + 2x_2 + x_3 = 10$$

$$2x_1 + 3x_2 + 2x_3 = 14$$

$$x_1 + 2x_2 + 3x_3 = 14$$

Solⁿ:

Coef^t matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Here,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Multiplying LU and setting the answers equal to A gives,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

1st Row,

$$u_{11} = 3$$

$$u_{12} = 2$$

$$u_{13} = 1$$

2nd Row,

$$l_{21}u_{11} = 2$$

$$l_{21} \times 3 = 2$$

$$l_{21} = 2/3$$

$$l_{21}u_{12} + u_{22} = 3$$

$$2/3 \times 2 + u_{22} = 3$$

$$u_{22} = 3 - 4/3$$

$$= 5/3$$

$$l_{21}u_{13} + u_{23} = 2$$

$$2/3 \times 1 + u_{23} = 2$$

$$u_{23} = 2 - 2/3$$

$$= 4/3$$

3rd Row,

$$l_{3,4,1} = 1$$

$$l_{3,1} \times 3 = 1$$

$$l_{3,1} = 1/3$$

$$l_{3,1} u_{1,2} + l_{3,2} u_{2,2} = 2$$

$$1/3 \times 2 + l_{3,2} \times 5/3 = 2$$

$$5/3 l_{3,2} = 2 - 2/3$$

$$5/3 l_{3,2} = 4/3$$

$$l_{3,2} = 4/3 \times 3/5$$

$$l_{3,2} = 4/5$$

$$l_{3,1} u_{1,3} + l_{3,2} u_{2,3} + u_{3,3} = 3$$

$$1/3 \times 1 + 4/5 \times 4/3 + u_{3,3} = 3$$

$$u_{3,3} = 3 - \frac{1}{3} - \frac{16}{15}$$

$$= \frac{45 - 5 - 16}{15}$$

$$= 24/15 = 8/5$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$$

Now, solving $[L][Z] = [C]$ using backsubstitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

we get,

$$\rightarrow Z_1 = 10$$

$$\rightarrow 2/3 Z_1 + Z_2 = 14$$

$$2/3 \times 10 + Z_2 = 14$$

$$Z_2 = 14 - \frac{20}{3} \\ = \frac{22}{3}$$

$$\rightarrow 1/3 Z_1 + 4/5 Z_2 + Z_3 = 14$$

$$1/3 \times 10 + 4/5 \times \frac{22}{3} + Z_3 = 14$$

$$Z_3 = \frac{24}{5}$$

again solve $[u][x] = [z]$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

$$\rightarrow 8/5 x_3 = 24/5$$

$$x_3 = 24/5 \times 5/8 \\ = 3$$

$$\rightarrow 5/3 x_2 + 4/3 x_3 = 22/3$$

$$5/3 x_2 + 4/3 \times 3 = 22/3$$

$$x_2 = \left(\frac{22}{3} - 4 \right) \times \frac{3}{5} \\ = 2$$

$$\rightarrow 3x_1 + 2x_2 + x_3 = 10$$

$$3x_1 + 2 \times 2 + 3 = 10$$

$$x_1 = 3/3$$

$$= 1$$

$$\text{Hence, } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} //$$

Q

$$\begin{aligned} 4x + y + 3z &= 11 \\ 3x + 4y + z &= 11 \\ 2x + 3y + z &= 7 \end{aligned}$$

Solⁿ:

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 11 \\ 11 \\ 7 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 3 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

Now,

Equating

1st Row

$$u_{11} = 4$$

$$u_{12} = 1$$

$$u_{13} = 3$$

2nd Row,

$$a_{21}u_{11} = 3$$

$$u_{21} = 3/4$$

$$a_{21}u_{12} + u_{22} = 4$$

$$u_{22} = 4 - 3/4 \times 2$$

$$u_{22} = 13/4$$

$$a_{21}u_{13} + u_{23} = 1$$

$$u_{23} = 1 - 3/4 \times 3$$

$$u_{23} = -5/4$$

3rd Row,

$$a_{31}u_{11} = 2$$

$$u_{31} = 1/2$$

$$a_{31}u_{12} + a_{32}u_{22} = 3$$

$$a_{32} \times 13/4 = 3 - \frac{1}{2} \times 2$$

$$a_{32} = 5/2 \times 4/13$$

$$a_{32} = 10/13$$

$$a_{31}u_{13} + a_{32}u_{23} + u_{33} = 1$$

$$u_{33} = 1 - \frac{1}{2} \times 3 - \frac{10}{13} \times (-5/4)$$

$$= 6/13$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ 1/2 & 10/13 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 \\ 0 & 13/4 & -5/4 \\ 0 & 0 & 6/13 \end{bmatrix}$$

Now,

$$LZ = C$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ 1/2 & 10/13 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \\ 7 \end{bmatrix}$$

Then,

$$z_1 = 11$$

$$3/4 z_1 + z_2 = 11$$

$$z_2 = 11 - 3/4 \times 11$$

$$= 11/4$$

$$1/2 z_1 + 10/13 z_2 + z_3 = 7$$

$$z_3 = 7 - 1/2 \times 11 - 10/13 \times 11/4$$

$$= -8/13$$

Here,

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 11/4 \\ -8/13 \end{bmatrix}$$

Then,

$$Ux = Z$$

$$\begin{bmatrix} 4 & 1 & 3 \\ 0 & 13/4 & -5/4 \\ 0 & 0 & 6/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 11/4 \\ -8/13 \end{bmatrix}$$

$$6/13 x_3 = -8/13$$

$$x_3 = -4/3$$

$$13/4 x_2 - 5/4 x_3 = 11/4$$

$$13/4 x_2 = 11/4 + 5/4 \times (-4/3)$$

$$x_2 = 1/3$$

$$4x_1 + x_2 + 3x_3 = 11$$

$$4x_1 = 11 - 1/3 - 3 \times (-4/3)$$

$$x_1 = 11/3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 1/3 \\ -4/3 \end{bmatrix}$$

cholesky method!

When coefficient matrix is symmetric, cholesky method is used.

If coefficient is symmetric then upper factor is transpose of lower factor or vice versa. Thus, we can factorize matrix A as,

$$A = [L][L^T] \text{ or } [U][U^T]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & 0 & \dots & 0 \\ u_{12} & u_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \dots & u_{nn} \end{bmatrix}$$

Then,

$$A = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11}^2 & u_{11}u_{12} & u_{11}u_{13} \\ u_{11}u_{12} & u_{12}^2 + u_{22}^2 & u_{12}u_{13} + u_{22}u_{23} \\ u_{11}u_{13} & u_{12}u_{13} + u_{22}u_{23} & u_{13}^2 + u_{23}^2 + u_{33}^2 \end{bmatrix}$$

Q Factorize the given matrix cholesky decomposition method.

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 80 & 44 \\ 7 & 44 & 89 \end{bmatrix}$$

Solⁿ

$$A = U U^T$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11}^2 & u_{11}u_{12} & u_{11}u_{13} \\ u_{11}u_{12} & u_{12}^2 + u_{22}^2 & u_{12}u_{13} + u_{22}u_{23} \\ u_{11}u_{13} & u_{12}u_{13} + u_{22}u_{23} & u_{13}^2 + u_{23}^2 + u_{33}^2 \end{bmatrix}$$

equating,

$$u_{11}^2 = 1$$

$$u_{11} = 1$$

$$u_{11}u_{12} = 4$$

$$u_{12} = 4$$

$$u_{11}u_{13} = 7$$

$$u_{13} = 7$$

$$U_{12}^2 + U_{22}^2 = 80$$

$$U_{22} = \sqrt{80 - U_{12}^2}$$

$$U_{22} = 8$$

$$U_{12}U_{13} + U_{22}U_{23} = 44$$

$$U_{23} = \frac{44 - 4 \times 7}{8}$$

$$U_{23} = 2$$

$$U_{13}^2 + U_{23}^2 + U_{33}^2 = 89$$

$$U_{33}^2 = 89 - 7^2 - 2^2$$

$$U_{33} = 6$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 8 & 0 \\ 7 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

//

Q

$$\begin{aligned} 4x_1 + 10x_2 + 8x_3 &= 44 \\ 10x_1 + 26x_2 + 26x_3 &= 128 \\ 8x_1 + 26x_2 + 61x_3 &= 214 \end{aligned}$$

solⁿ

$$A = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \quad C = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$$A = UTU$$

$$= \begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} U_{11}^2 & U_{11}U_{12} & U_{11}U_{13} \\ U_{11}U_{12} & U_{12}^2 + U_{22}^2 & U_{12}U_{13} + U_{22}U_{23} \\ U_{11}U_{13} & U_{13}U_{12} + U_{23}U_{22} & U_{13}^2 + U_{23}^2 + U_{33}^2 \end{bmatrix}$$

Equating

$$U_{11}^2 = 4$$

$$U_{11} = 2$$

$$U_{11}U_{12} = 10$$

$$U_{12} = 5$$

$$U_{11}U_{13} = 8$$

$$U_{13} = 4$$

$$U_{12}^2 + U_{22}^2 = 26$$

$$U_{12}U_{13} + U_{22}U_{23} = 26$$

$$U_{22}^2 = 1$$

$$1 \times U_{23} = 26 - 5 \times 4$$

$$U_{22} = 1$$

$$U_{23} = 6$$

$$U_{11}^2 + U_{22}^2 + U_{33}^2 = 61$$

$$U_{33}^2 = 61 - 16 - 36 = 9$$

$$U_{33} = 3$$

$$A = U^T U = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Now,

$$U^T Z = C$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

Equating,

$$2z_1 = 44$$

$$z_1 = 22$$

$$5z_1 + z_2 = 128$$

$$z_2 = 128 - 5 \times 22$$

$$= 18$$

$$4z_1 + 6z_2 + 3z_3 = 214$$

$$3z_3 = 214 - 4 \times 22 - 6 \times 18$$

$$z_3 = 6$$

Then,

$$UX = Z$$

$$\begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

$$x_3 = 2$$

$$x_2 + 6x_3 = 18$$

$$x_2 = 6$$

$$X = \begin{bmatrix} -8 \\ 6 \\ 2 \end{bmatrix}$$

$$2x_1 + 5x_2 + 4x_3 = 22$$

$$x_1 = -8$$

Iteration Method for solving system of linear equation:

i. The amount of storage space required for iterative solution on a computer is far less than the one required for direct method.

ii) Due to round off errors, direct method becomes less efficient than iterative methods.

1) Jacobi iteration method:

It is used to solving a matrix equation. In a matrix there has no zeros along the main diagonal of coefficient matrix. Let us consider, the system of linear equation in n -unknown.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

We can write the original system as,

$$x_1 = \frac{b_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + \dots + a_{2n}x_n)}{a_{22}}$$

$$x_n = \frac{b_n - (a_{n1}x_1 + \dots + a_{nn-1}x_{n-1})}{a_{nn}}$$

- Compute $x_1, x_2, \dots, x_n$ by using initial guess for that values.
- Computed values are again used to compute the next set of x -values, the process can be continued till we obtained the desired level of accuracy in the x values.
- At the end of each iteration, the absolute relative approximate error for each x is calculated as,

$$|E_v| = \left| \frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}} \right|$$

where,

x_i^{new} is the recently obtained value of x_i as x_i^{old} is the previous value of x_i .

Q Use the Jacobi iteration method to obtain the solution of the following equations:

$$6x_1 - 2x_2 + x_3 = 11$$

$$x_1 + 2x_2 - 5x_3 = -1$$

$$-2x_1 + 7x_2 + 2x_3 = 5$$

Solⁿ:

Re-write the equations such that each equation has the unknown with largest coefficient on the left hand side.

$$x_1 = \frac{11 + 2x_2 - x_3}{6}$$

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7}$$

$$x_3 = \frac{x_1 + 2x_2 + 1}{5}$$

Iteration - 1

Assume the initial guesses $(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$, then calculate $(x_1)^1$, $(x_2)^1$ and $(x_3)^1$.

$$(x_1)^1 = \frac{2(x_2)^0 - (x_3)^0 + 11}{6} = \frac{2 \times 0 - 0 + 11}{6} = 1.833$$

$$(x_2)^1 = \frac{2(x_1)^0 - 2(x_3)^0 + 5}{7} = \frac{2 \times 0 - 2 \times 0 + 5}{7} = 0.714$$

$$(x_3)^1 = \frac{(x_1)^0 + 2(x_2)^0 + 1}{5} = \frac{0 + 2 \times 0 + 1}{5} = 0.200$$

Iteration - 2 :

Use the obtained value in the 1st iteration, to calculate the values for 2nd iteration.

$$(x_1)^2 = \frac{2(x_2)^1 - (x_3)^1 + 11}{6} = \frac{2 \times 0.714 - 0.2 + 11}{6} = 2.038$$

$$(x_2)^2 = \frac{2(x_1)^1 - 2(x_3)^1 + 5}{7} = \frac{2 \times 1.833 - 2 \times 0.2 + 5}{7} = 1.185$$

$$(x_3)^2 = \frac{(x_1)^1 + 2(x_2)^1 + 1}{5} = \frac{1.833 + 2 \times 0.714 + 1}{5} = 0.852$$

Iteration - 3 :

Use the obtained value in the 2nd iteration, to calculate the values for 3rd iteration.

$$(\lambda_1)^3 = \frac{11 + 2(\lambda_2)^2 - (\lambda_3)^2}{6} = \frac{11 + 2 \times 1.181 - 0.852}{6}$$

$$= 2.004$$

$$(\lambda_2)^3 = \frac{2(\lambda_1)^2 - 2(\lambda_3)^2 + 5}{7} = \frac{2 \times 2.038 - 2 \times 0.852 + 5}{7}$$

$$= 1.001$$

$$(\lambda_3)^3 = \frac{(\lambda_1)^2 + 2(\lambda_2)^2 + 1}{5} = \frac{2.038 + 2 \times 1.181 + 1}{5}$$

$$= 1.038$$

4th iteration,

$$(\lambda_1)^4 = \frac{11 + 2(\lambda_2)^3 - (\lambda_3)^3}{6} = \frac{11 + 2 \times 1.001 - 1.038}{6}$$

$$= 1.994$$

$$(\lambda_2)^4 = \frac{2(\lambda_1)^3 - 2(\lambda_3)^3 + 5}{7} = \frac{2 \times 2.004 - 2 \times 1.038 + 5}{7}$$

$$= 0.990$$

$$(\lambda_3)^4 = \frac{(\lambda_1)^3 + 2(\lambda_2)^3 + 1}{5} = \frac{2.004 + 2 \times 1.001 + 1}{5}$$

$$= 1.001$$

5th iteration:

$$(x_1)^5 = \frac{11 + 2(x_2)^4 - (x_3)^4}{6} = \frac{11 + 2 + 0.990 - 1.001}{6} \\ = 1.997$$

$$(x_2)^5 = \frac{2(x_1)^4 - 2(x_3)^4 + 5}{7} = \frac{2 \times 1.994 - 2 \times 1.001 + 5}{7} \\ = 0.998$$

$$(x_3)^5 = \frac{(x_1)^4 + 2(x_2)^4 + 1}{5} = \frac{1.994 + 2 \times 0.990 + 1}{5} \\ = 0.999$$

6th iteration:

$$(x_1)^6 = \frac{11 + 2(x_2)^5 - (x_3)^5}{6} = \frac{11 + 2 \times 0.998 - 0.999}{6} \\ = 1.999$$

$$(x_2)^6 = \frac{2(x_1)^5 - 2(x_3)^5 + 5}{7} = \frac{2 \times 1.997 - 2 \times 0.999 + 5}{7} \\ = 0.999$$

$$(x_3)^6 = \frac{(x_1)^5 + 2(x_2)^5 + 1}{5} = \frac{1.997 + 2 \times 0.998 + 1}{5} \\ = 0.999$$

7th iteration:

$$(x_1)^7 = \frac{11 + 2(x_2)^6 - (x_3)^6}{6} = \frac{11 + 2 \times 0.999 - 0.999}{6}$$

$$= 1.9998$$

$$(x_2)^7 = \frac{2(x_1)^6 - 2(x_3)^6 + 5}{7} = \frac{2 \times 1.999 - 2 \times 0.999 + 5}{7}$$

$$= 1.000$$

$$(x_3)^7 = \frac{(x_1)^6 + 2(x_2)^6 + 1}{5} = \frac{1.999 + 2 \times 0.999 + 1}{5}$$

$$= 0.9994$$

8th iteration:

$$(x_1)^8 = \frac{11 + 2(x_2)^7 - (x_3)^7}{6} = \frac{11 + 2 \times 1.000 - 0.9994}{6}$$

$$= 2.000$$

$$(x_2)^8 = \frac{2(x_1)^7 - 2(x_3)^7 + 5}{7} = \frac{2 \times 1.9998 - 2 \times 0.9994 + 5}{7}$$

$$= 1.000$$

$$(x_3)^8 = \frac{(x_1)^7 + 2(x_2)^7 + 1}{5} = \frac{1.9998 + 2 \times 1.000 + 1}{5}$$

$$= 0.9999$$

g^{th} iteration:

$$(x_1)^g = \frac{11 + 2(x_2)^8 - (x_3)^8}{6} = \frac{11 + 2 \times 1 - 0.9999}{6}$$
$$= 2.000$$

$$(x_2)^g = \frac{2(x_1)^8 - 2(x_3)^8 + 5}{7} = \frac{2 \times 2 - 2 \times 0.9999 + 5}{7}$$
$$= 1.000$$

$$(x_3)^g = \frac{(x_1)^8 + 2(x_2)^8 + 1}{5} = \frac{2 + 2 \times 1 + 1}{5}$$
$$= 1.000$$

2) Gauss Seidal Method:

- It is improved version of Jacobi iteration method.
- In JIM, we obtain with initial values $(x_1)^0, (x_2)^0, \dots, (x_n)^0$ and $(x_1)^1, (x_2)^1, \dots, (x_n)^1$.
- The Gauss Seidal method uses the most recent value of x as soon as they become available at any point of iteration process.

9

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ x_1 + 2x_2 - 5x_3 &= -2 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \end{aligned}$$

sol:

$$x_1 = \frac{2x_2 - x_3 + 11}{6}$$

$$x_2 = \frac{2x_1 - 7x_3 + 5}{7}$$

$$x_3 = \frac{x_1 + 2x_2 + 2}{5}$$

Iteration - 1:

Assume the initial guesses $(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$, then calculate $(x_1)^1$:

$$(x_1)^1 = \frac{2(x_2)^0 - (x_3)^0 + 11}{6} = \frac{2(0) - (0) + 11}{6} = 1.833$$

Use $(x_1)^1 = 1.833$ and $(x_3)^0$ to calculate $(x_2)^1$.

$$(x_2)^1 = \frac{2(x_1)^1 - 2(x_3)^0 + 5}{7} = \frac{2 \times (1.833) - 2(0) + 5}{7} \\ = 1.238$$

Similarly, Use $(x_1)^1 = 1.833$ and $(x_2)^1 = 1.238$ to calculate $(x_3)^1$.

$$(x_3)^1 = \frac{(x_1)^1 + 2(x_2)^1 + 1}{5} = \frac{1.833 + 2 \times 1.238 + 1}{5} \\ = 1.062$$

Iteration 2:

$$(x_1)^2 = \frac{2(x_2)^1 - (x_3)^1 + 11}{6} = \frac{2 \times 1.238 - 1.062 + 11}{6} \\ = 2.069$$

$$(x_2)^2 = \frac{2(x_1)^2 - 2(x_3)^1 + 5}{7} = \frac{2 \times 2.069 - 2 \times 1.062 + 5}{7} \\ = 1.002$$

$$(x_3)^2 = \frac{(x_1)^2 + 2(x_2)^2 + 1}{5} = \frac{2.069 + 2 \times 1.002 + 1}{5} \\ = 1.015$$

Iteration - 3:

$$(x_1)^3 = \frac{2(x_2)^2 - (x_3)^2 + 11}{6} = \frac{2 \times 1.002 - 1.015 + 11}{6} \\ = 1.998$$

$$(x_2)^3 = \frac{2(x_1)^3 - 2(x_3)^2 + 5}{7} = \frac{2 \times 1.998 - 2 \times 1.015 + 5}{7} \\ = 0.995$$

$$(x_3)^3 = \frac{(x_1)^3 + 2(x_2)^3 + 1}{5} = \frac{1.998 + 2 \times 0.995 + 1}{5} \\ = 0.998$$

Iteration - 4:

$$(x_1)^4 = \frac{2(x_2)^3 - (x_3)^3 + 11}{6} = \frac{2 \times 0.995 - 0.998 + 11}{6} \\ = 1.999$$

$$(x_2)^4 = \frac{2(x_1)^4 - 2(x_3)^3 + 5}{7} = \frac{2 \times 1.999 - 2 \times 0.998 + 5}{7} \\ = 1.000$$

$$(x_3)^4 = \frac{(x_1)^4 + 2(x_2)^4 + 1}{5} = \frac{1.999 + 2 \times 1 + 1}{5} \\ = 1.000$$

Iteration 5:

$$(x_1)^5 = \frac{2(x_2)^4 - (x_3)^4 + 11}{6} = \frac{2*1 - 1 + 11}{6} \\ = 2.000$$

$$(x_2)^5 = \frac{2(x_1)^5 - 2(x_3)^4 + 5}{7} = \frac{2*2 - 2*1 + 5}{7} \\ = 1.000$$

$$(x_3)^5 = \frac{(x_1)^5 + 2(x_2)^5 + 1}{5} = \frac{2 + 2*1 + 1}{5} \\ = 1.000$$

Eigen Value and eigen Vector problem:

- Let A be the square matrix of order $n \times n$, then a number λ is said to be eigenvalue of matrix A if there exist a column matrix X of $n \times 1$. Such that,

$$AX = \lambda X$$

$\swarrow$ Eigen vector.
 $\searrow$ Eigenvalue

If the problem can be reduced to the system of equation of the form $AX = \lambda X$ such problems are called eigenvalue problem.

Power Method:

- It is used to find out eigenvalue and eigenvectors of a large matrices.
- Power method is a single value method used for determining the largest eigen value and vector of a matrix.
- The method fails when there is two or more largest eigen values.
- It is a iteration process using a starting vector. Power method can be implemented as follows:

$$Y = AX \quad \text{--- (1)}$$

$$X = \frac{1}{k} Y \quad \text{--- (2)}$$

y is calculated from equation 1st and now x is calculated using equation 2nd. K is a scaling factor, which is element of y with largest magnitude.

$$Q \quad \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solⁿ

Assume the initial guess for eigenvector as.

$$X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Iteration 1st:

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow K = 2 \quad [\because \text{Largest value of } Y]$$

New value of X can be calculated as,

$$X = \frac{1}{K} Y = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

Iteration 2nd

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix}$$

$$\Rightarrow K = 2.5$$

$$X = \frac{1}{K} Y = \frac{1}{2.5} \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

Iteration 3rd

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix}$$

$$\Rightarrow K = 2.8$$

$$X = \frac{1}{2.8} \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix}$$

Iteration 4th

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix}$$

$$2) k = 2.93$$

$$x = \frac{1}{2.93} \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix}$$

$$z = \begin{bmatrix} 0.976 \\ 1 \\ 0 \end{bmatrix}$$

Iteration 5th!

$$y = Ax = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.976 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.976 \\ 2.952 \\ 0 \end{bmatrix}$$

$$\Rightarrow k = 2.976$$

$$x = \frac{1}{2.976} \begin{bmatrix} 2.976 \\ 2.952 \\ 0 \end{bmatrix}$$

$$z = \begin{bmatrix} 1 \\ 0.992 \\ 0 \end{bmatrix}$$

Iteration 6 :

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.992 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.984 \\ 2.992 \\ 0 \end{bmatrix}$$

$$\therefore K = 2.992$$

$$X = \frac{1}{2.992} \begin{bmatrix} 2.984 \\ 2.992 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.997 \\ 1 \\ 0 \end{bmatrix}$$

Iteration 7 :

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.96 \\ 0 \end{bmatrix}$$

$$\therefore K = 2.98$$

$$X = \frac{1}{2.98} \begin{bmatrix} 2.98 \\ 2.96 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (\because \text{Round off})$$

$$= \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (\text{Round off})$$

Eigen Value = Largest value of γ

$$(\lambda) = 3.$$

Eigen Vector

$$X.2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} //$$

Unit - 6

Solution of Partial Differential equations.

Disclaimer:

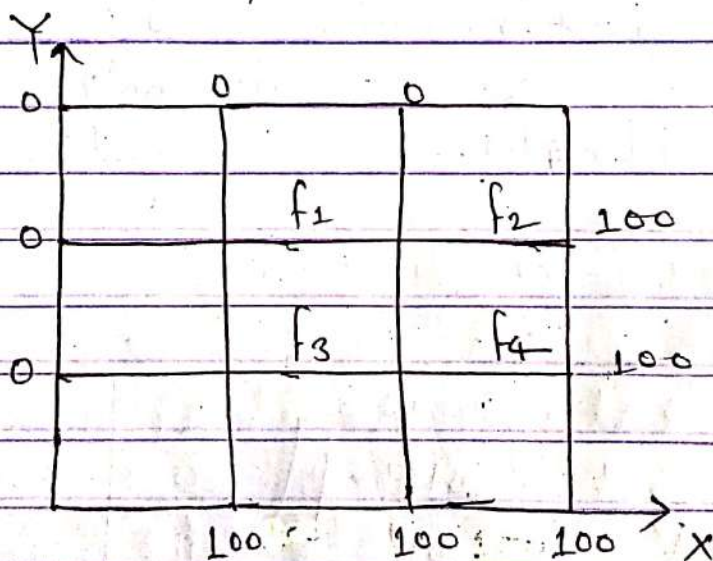
This chapter is very important
for long question. But,
we can do only example
from this chapter.

1 >> Solving Laplace's Equation:

eg: A steel plate of size 30×30 cm is given. Two adjacent sides are placed at 100°C and other sides are held at 0°C . Find the temperature at interior points assuming the grid size of 10×10 cm.

Solⁿ:

A steel plate is shown in figure below:



At point b_1 :

$$0 + 0 + f_2 + f_3 - 4b_1 = 0$$

$$\text{or, } -4b_1 + b_2 + b_3 = 0 \quad \text{--- (i)}$$

At point b_2 :

$$0 + 100 + b_4 - 4b_2 + b_1 = 0$$

$$b_1 - 4b_2 + b_4 = -100 \text{ --- (ii)}$$

At point b_3 :

$$0 + 100 + b_1 + b_4 - 4b_3 = 0$$

$$b_1 - 4b_3 + b_4 = -100 \text{ --- (iii)}$$

At point b_4 :

$$100 + 100 + b_2 + b_3 - 4b_4 = 0$$

$$b_2 + b_3 - 4b_4 = -200 \text{ --- (iv)}$$

Arranging above equations,

$$-4b_1 + b_2 + b_3 = 0 \text{ --- (a)}$$

$$b_1 - 4b_2 + b_4 = -100 \text{ --- (b)}$$

$$b_1 - 4b_3 + b_4 = -100 \text{ --- (c)}$$

$$b_2 + b_3 - 4b_4 = -200 \text{ --- (d)}$$

Now,

the adjacent matrix is,

$$\begin{bmatrix} -4 & 1 & 1 & 0 & : & 0 \\ 1 & -4 & 0 & 1 & : & -100 \\ 1 & 0 & -4 & 1 & : & -100 \\ 0 & 1 & 1 & -4 & : & -200 \end{bmatrix}$$

$$R_1 \rightarrow -\frac{1}{4} R_1$$

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 & : & 0 \\ 1 & -4 & 0 & 1 & : & -100 \\ 1 & 0 & -4 & 1 & : & -100 \\ 0 & 1 & 1 & -4 & : & -200 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \text{ \& } R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 & : & 0 \\ 0 & -15/4 & 1/4 & 1 & : & -100 \\ 0 & 1/4 & -15/4 & 1 & : & -100 \\ 0 & 1 & 1 & -4 & : & -200 \end{bmatrix}$$

$$R_2 \rightarrow -4/15 R_2$$

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 & 0 \\ 0 & 1 & -1/15 & -4/15 & 80/3 \\ 0 & 1/4 & -15/4 & 1 & -100 \\ 0 & 1 & 1 & -4 & -200 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 1/4 R_2, R_3 \rightarrow R_3 - 1/4 R_2 \text{ \& } R_4 \rightarrow R_4 - R_2$$

$$\begin{bmatrix} 1 & 0 & -4/15 & -1/15 & 20/3 \\ 0 & 1 & -1/15 & -4/15 & 80/3 \\ 0 & 0 & -56/15 & 16/15 & -320/3 \\ 0 & 0 & 16/15 & -56/15 & -680/3 \end{bmatrix}$$

$$R_3 \rightarrow -15/56 R_3$$

$$\begin{bmatrix} 1 & 0 & -4/15 & -1/15 & 20/3 \\ 0 & 1 & -1/15 & -4/15 & 80/3 \\ 0 & 0 & 1 & -24/7 & 200/7 \\ 0 & 0 & 16/15 & -56/15 & -680/3 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 4/15 R_3, R_2 \rightarrow R_2 + 1/15 R_3 \text{ \& } R_4 \rightarrow R_4 - 16/15 R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -1/7 & 100/7 \\ 0 & 1 & 0 & -2/7 & 200/7 \\ 0 & 0 & 1 & -2/7 & 200/7 \\ 0 & 0 & 0 & -24/7 & -1800/7 \end{bmatrix}$$

$$R_4 \rightarrow -7/24 R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & -1/7 & : & 100/7 \\ 0 & 1 & 0 & -2/7 & : & 200/7 \\ 0 & 0 & 1 & -2/7 & : & 200/7 \\ 0 & 0 & 0 & 1 & : & 75 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 1/7 R_4, R_2 \rightarrow R_2 + 2/7 R_4 \text{ \& } R_3 \rightarrow R_3 + 2/7 R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 25 \\ 0 & 1 & 0 & 0 & : & 50 \\ 0 & 0 & 1 & 0 & : & 50 \\ 0 & 0 & 0 & 1 & : & 75 \end{bmatrix}$$

Here,

$$b_1 = 25$$

$$b_2 = 50$$

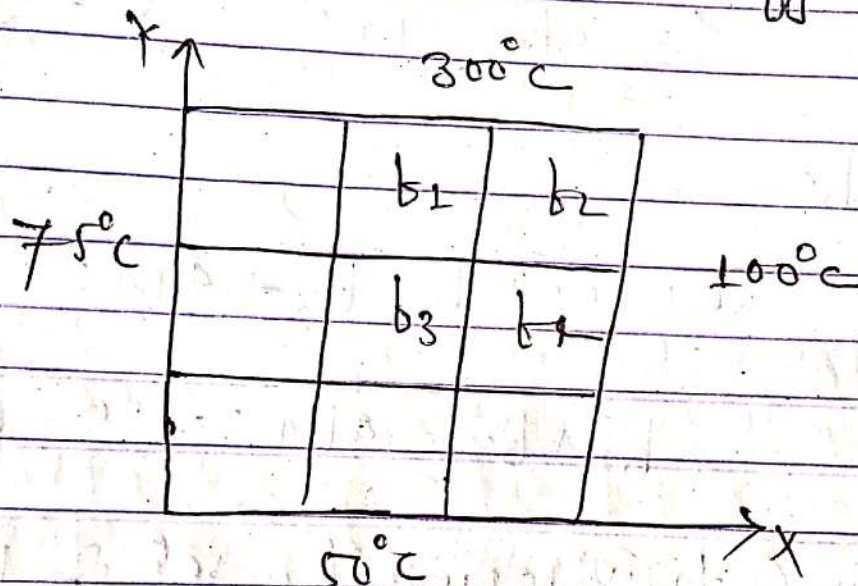
$$b_3 = 50$$

$$b_4 = 75$$

//

eg. A plate of dimension $2.4\text{ m} \times 2.4\text{ m}$ is subjected to temperatures as follows: left side at 75°C , right side at 100°C , upper part at 300°C , and lower side at 50°C . If square grid length of $0.8\text{ m} \times 0.8\text{ m}$ is assumed, what will be temperature at the interior nodes.

The nodes are shown in figure below,



Now, to get the temperature at the interior nodes we have to write equation for all the combinations of are,

for b_1 :

$$300 + 75 + b_2 + b_3 - 4b_1 = 0$$

$$-4b_1 + b_2 + b_3 = -375 \quad \text{--- (1)}$$

for b_2 :

$$300 + 200 + b_1 + b_4 - 4b_2 = 0$$

$$b_1 - 4b_2 + b_4 = -400 \text{ --- (ii)}$$

for b_3 :

$$75 + 50 + b_1 + b_4 - 4b_3 = 0$$

$$b_1 - 4b_3 + b_4 = -125 \text{ --- (iii)}$$

for b_4 :

$$50 + 100 + b_2 + b_3 - 4b_4 = 0$$

$$b_2 + b_3 - 4b_4 = -150 \text{ --- (iv)}$$

Equation (i) to (iv) represent a set of linear equations

$$-4b_1 + b_2 + b_3 = -375 \text{ --- (a)}$$

$$b_1 - 4b_2 + b_4 = -400 \text{ --- (b)}$$

$$b_1 - 4b_3 + b_4 = -125 \text{ --- (c)}$$

$$b_2 + b_3 - 4b_4 = -150 \text{ --- (d)}$$

Now, the adjacent forms is,

$$\begin{bmatrix} -4 & 1 & 1 & 0 & : & -375 \\ 1 & -4 & 0 & 1 & : & -400 \\ 1 & 0 & -4 & 1 & : & -125 \\ 0 & 1 & 1 & -4 & : & -150 \end{bmatrix}$$

$$R_1 \rightarrow -\frac{1}{4}R_1$$

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 & : & -375/4 \\ 1 & -4 & 0 & 1 & : & -400 \\ 1 & 0 & -4 & 1 & : & -125 \\ 0 & 1 & 1 & -4 & : & -150 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \text{ \& } R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 & : & -375/4 \\ 0 & -15/4 & 1/4 & 1 & : & -1225/4 \\ 0 & 1/4 & -15/4 & 1 & : & -125/4 \\ 0 & 1 & 1 & -4 & : & -150 \end{bmatrix}$$

$$R_2 \rightarrow -4/15 R_2$$

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 & : & -375/4 \\ 0 & 1 & -1/15 & -4/15 & : & 245/3 \\ 0 & 1/4 & -15/4 & 1 & : & -125/4 \\ 0 & 1 & 1 & -4 & : & -150 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + \frac{1}{4}R_2, R_3 \rightarrow R_3 - \frac{1}{4}R_2 \text{ \& } R_4 \rightarrow R_4 - R_2$$

$$\begin{bmatrix} 1 & 0 & -4/15 & -1/15 & : & -220/3 \\ 0 & 1 & -1/15 & -4/15 & : & 245/3 \\ 0 & 0 & -56/15 & 16/15 & : & -155/3 \\ 0 & 0 & 16/15 & -56/15 & : & -695/3 \end{bmatrix}$$

$$R_3 \rightarrow -15/56 R_3$$

$$\begin{bmatrix} 1 & 0 & -4/15 & -1 & : & -220/3 \\ 0 & 1 & -1/15 & -4/15 & : & 245/3 \\ 0 & 0 & 1 & -2/7 & : & 775/56 \\ 0 & 0 & 16/15 & -56/15 & : & -695/3 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 4/15 R_3, R_2 \rightarrow R_2 + 1/15 R_3 \text{ \& } R_4 \rightarrow R_4 - 16/15 R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -4/7 & : & -975/14 \\ 0 & 1 & 0 & -2/7 & : & 4625/56 \\ 0 & 0 & 1 & -2/7 & : & 775/56 \\ 0 & 0 & 0 & -24/7 & : & -1725/7 \end{bmatrix}$$

$$R_4 \rightarrow -7/24 R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & -4/7 & : & -975/14 \\ 0 & 1 & 0 & -2/7 & : & 4625/56 \\ 0 & 0 & 1 & -2/7 & : & 775/56 \\ 0 & 0 & 0 & 1 & : & 575/8 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 4/7 R_4, R_2 \rightarrow R_2 + 2/7 R_4 \text{ \& } R_3 \rightarrow R_3 + 2/7 R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & -475/8 \\ 0 & 1 & 0 & 0 & : & 825/8 \\ 0 & 0 & 1 & 0 & : & 275/8 \\ 0 & 0 & 0 & 1 & : & 575/8 \end{bmatrix}$$

here,

$$b_1 = -475/8$$

$$b_2 = 825/8$$

$$b_3 = 275/8$$

$$b_4 = 575/8$$

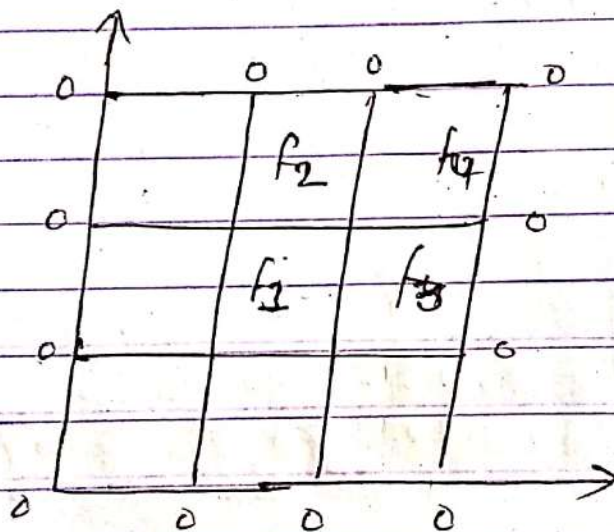
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2. Solving Poisson's Equation.

eg: solve the Poisson's equation $\nabla^2 f = 2xy^2$ over the square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$ with $f=0$ on the boundary and $h=1$.

Sol:

The divide the domain into grids as below:



At point f_1 :

$$f_2 + f_3 - 4f_1 = 2 \times 1^2 \times 1^2 = 2$$

At point f_2 :

$$f_1 + f_4 - 4f_2 = 2 \times 2^2 \times 1^2 = 8$$

At point f_3 :

$$f_1 + f_4 - 4f_3 = 2 \times 1^2 \times 2^2 = 8$$

At point f_4 :

$$f_2 + f_3 - 4f_4 = 2 \times 2^2 \times 2^2 = 32$$

Now, we have following system of equations;

$$f_2 + f_3 - 4f_1 = 2$$

$$4f_1 = \frac{f_2 + f_3 + 2}{4}$$

$$f_1 + f_4 - 4f_2 = 8$$

$$f_2 = \frac{f_1 + f_4 - 8}{4}$$

$$f_1 + f_4 - 4f_3 = 8$$

$$f_3 = \frac{f_1 + f_4 - 8}{4}$$

$$f_2 + f_3 - 4f_4 = 32$$

$$f_4 = \frac{f_2 + f_3 - 32}{4}$$

Solving above system of equations by using Gauss-Seidel method, we get

Iteration 1st,

$$f_1 = \frac{f_2 + f_3 - 2}{4} = \frac{0 + 0 - 2}{4} = -0.5$$

$$f_2 = \frac{f_1 + f_4 - 8}{4} = \frac{-0.5 + 0 - 8}{4} = -2.125$$

$$f_3 = \frac{f_1 + f_4 - 8}{4} = \frac{-0.5 + 0 - 8}{4} = -2.125$$

$$f_4 = \frac{f_2 + f_3 - 32}{4} = \frac{-2.125 - 2.125 - 32}{4} = -9.0625$$

Iteration 2nd,

$$f_1 = \frac{-2.125 - 2.125 - 2}{4} = -1.5625$$

$$f_2 = \frac{-1.5625 - 9.0625 - 8}{4} = -4.65625$$

$$f_3 = \frac{-1.5625 - 9.0625 - 8}{4} = -4.65625$$

$$f_4 = \frac{-4.65625 - 4.65625 - 32}{4} = -10.328125$$

Iteration 3rd,

$$f_1 = \frac{-4.656 - 4.656 - 2}{4} = -2.828$$

$$f_2 = \frac{-2.828 - 10.328 - 8}{4} = -5.289$$

$$f_3 = \frac{-2.828 - 10.328 - 8}{4} = -5.289$$

$$f_4 = \frac{-5.289 - 5.289 - 32}{4} = -10.645$$

Iteration 4th,

$$f_1 = \frac{-5.289 - 5.289 - 2}{4} = -3.145$$

$$f_2 = \frac{-3.145 - 10.645 - 8}{4} = -5.447$$

$$f_3 = \frac{-3.145 - 10.645 - 8}{4} = -5.447$$

$$f_4 = \frac{-5.447 - 5.447 - 32}{4} = -10.724$$

Iteration 5th,

$$f_1 = \frac{-5.447 - 5.447 - 2}{4} = -3.224$$

$$f_2 = \frac{-3.224 - 10.724 - 8}{4} = -5.487$$

$$f_3 = \frac{-3.224 - 10.724 - 8}{4} = -5.487$$

$$f_4 = \frac{-5.487 - 5.487 - 32}{4} = -10.743$$

Iteration 6th;

$$f_1 = \frac{-5.487 - 5.487 - 2}{4} = -3.243$$

$$f_2 = \frac{-3.243 - 10.743 - 8}{4} = -5.497$$

$$f_3 = \frac{-3.243 - 10.743 - 8}{4} = -5.497$$

$$f_4 = \frac{-5.497 - 5.497 - 32}{4} = -10.748$$

Iteration 7th,

$$f_1 = \frac{-5.497 - 5.497 - 2}{4} = -3.248$$

$$f_2 = \frac{-3.248 - 10.748 - 8}{4} = -5.499$$

$$f_3 = \frac{-3.248 - 10.748 - 8}{4} = -5.499$$

$$f_4 = \frac{-5.499 - 5.499 - 32}{4} = -10.750$$

Iteration 8th,

$$f_1 = \frac{-5.499 - 5.499 - 2}{4} = -3.250$$

$$f_2 = \frac{-3.250 - 10.750 - 8}{4} = -5.500$$

$$f_3 = \frac{-3.250 - 10.750 - 8}{4} = -5.500$$

$$f_4 = \frac{-5.500 - 5.500 - 32}{4} = -10.750$$

Iteration g^{th} ,

$$f_1 = \frac{-5.500 - 5.500 - 2}{4} = -3.250$$

$$f_2 = \frac{-3.250 - 10.750 - 8}{4} = -5.500$$

$$f_3 = \frac{-3.250 - 10.750 - 8}{4} = -5.500$$

$$f_4 = \frac{-5.500 - 5.500 - 32}{4} = -10.750$$

Thus,

$$f_1 = -3.25$$

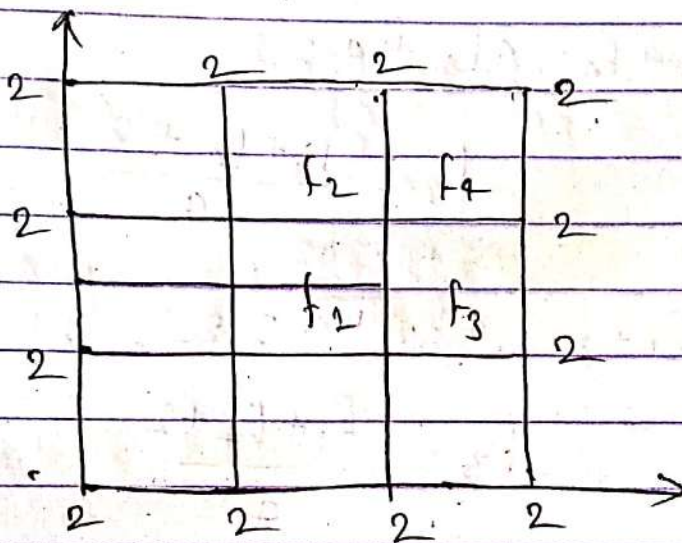
$$f_2 = -5.5$$

$$f_3 = -5.5$$

$$f_4 = -10.75$$

Q Solve the poisson's equation $\nabla^2 f = f(x,y)$ with $f(x,y) = xy$ and $f = 2$ on boundary by assuming square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$ and $h = 1$

Let's divide the domain into grids below:



For b_1 :

$$f_2 + f_3 + 4f_1 + 4 = 1 \times 1 = 1 \quad \text{--- (i)}$$

for b_2 ,

$$f_1 + f_4 + 4f_2 + 4 = 2 \times 1 = 2 \quad \text{--- (ii)}$$

for b_3 ,

$$f_1 + f_4 + 4f_3 + 4 = 1 \times 2 = 2 \quad \text{--- (iii)}$$

for b_4 ,

$$f_2 + f_3 + 4f_4 + 4 = 2 \times 2 = 4 \quad \text{--- (iv)}$$

Now, we have following system of equations,

$$f_2 + f_3 - 4f_1 + 4 = 2$$

$$f_1 = \frac{f_2 + f_3 + 3}{4}$$

$$f_1 + f_4 - 4f_2 + 4 = 2$$

$$f_2 = \frac{f_1 + f_4 + 2}{4}$$

$$f_1 + f_4 - 4f_3 + 4 = 2$$

$$f_3 = \frac{f_1 + f_4 + 2}{4}$$

$$f_2 + f_3 - 4f_4 + 4 = 4$$

$$f_4 = \frac{f_2 + f_3}{4}$$

Solving above system of equations by using Gauss-Seidel Method,

Iteration 1st,

$$f_1 = \frac{f_2 + f_3 + 3}{4} = \frac{0 + 0 + 3}{4} = \frac{3}{4} = 0.750$$

$$f_2 = \frac{f_1 + f_4 + 2}{4} = \frac{0.750 + 0 + 2}{4} = \frac{2.750}{4} = 0.688$$

$$f_3 = \frac{f_1 + f_4 + 2}{4} = \frac{0.750 + 0 + 2}{4} = 0.688$$

$$f_4 = \frac{f_2 + f_3}{4} = \frac{0.688 + 0.688}{4} = 0.344$$

Iteration 2nd,

$$f_1 = \frac{0.688 + 0.688 + 3}{4} = 1.094$$

$$f_2 = \frac{1.094 + 0.344 + 2}{4} = 0.859$$

$$f_3 = \frac{1.094 + 0.344 + 2}{4} = 0.859$$

$$f_4 = \frac{0.859 + 0.859}{4} = 0.430$$

Iteration 3rd,

$$f_1 = \frac{0.859 + 0.859 + 3}{4} = 1.180$$

$$f_2 = \frac{1.180 + 0.430 + 2}{4} = 0.902$$

$$f_3 = \frac{1.180 + 0.430 + 2}{4} = 0.902$$

$$f_4 = \frac{0.902 + 0.902}{4} = 0.451$$

Iteration 4th,

$$f_1 = \frac{0.902 + 0.902 + 3}{4} = 1.201$$

$$f_2 = \frac{1.201 + 0.451 + 2}{4} = 0.913$$

$$f_3 = \frac{1.201 + 0.451 + 2}{4} = 0.913$$

$$f_4 = \frac{0.913 + 0.913}{4} = 0.457$$

Iteration 5th,

$$f_1 = \frac{0.913 + 0.913 + 3}{4} = 1.207$$

$$f_2 = \frac{1.207 + 0.457 + 2}{4} = 0.916$$

$$f_3 = \frac{1.207 + 0.457 + 2}{4} = 0.916$$

$$f_4 = \frac{0.916 + 0.916}{4} = 0.458$$

Iteration 6th,

$$f_1 = \frac{0.916 + 0.916 + 3}{4} = 1.208$$

$$f_2 = \frac{1.208 + 0.458 + 2}{4} = 0.916$$

$$f_3 = \frac{1.208 + 0.458 + 2}{4} = 0.916$$

$$f_4 = \frac{0.916 + 0.916}{4} = 0.458$$

Iteration 7th,

$$f_1 = \frac{0.916 + 0.916 + 3}{4} = 1.208$$

$$f_2 = \frac{1.208 + 0.458 + 2}{4} = 0.916$$

$$f_3 = \frac{1.208 + 0.458 + 2}{4} = 0.916$$

$$f_4 = \frac{0.916 + 0.916 + 2}{4} = 0.458$$

Thus,

$$f_1 = 1.208$$

$$f_2 = 0.916$$

$$f_3 = 0.916$$

$$f_4 = 0.458$$

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